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Issues on Integrating Real and Financial Decisions

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Abstract:
We study the issue of integrating real and financial decisions in the monopoly framework. To that end, we combine the decisions of the firm with the decisions of the shareholders. When the managing shareholder chooses production, risk allocation, and the total number of shares for the risky asset, we show that there is no Nash equilibrium with a competitive financial market. Existence is reestablished under various restrictions on the set for the total number of shares. Moreover, there exists a Stackelberg equilibrium when the managing shareholder is the leader. In addition to discussing the issue of existence, we compare the equilibrium outcomes for each restriction we impose.

Keywords: Existence of Equilibrium, Financial sector, Firm behavior, Market power, Monopoly, Nash equilibrium, Perfect competition, Publicly-traded firm, Risk aversion, Risk taking, Shareholder behavior, Stackelberg equilibrium

JEL Classification: D21, D42, D82, D83, D84, L12, L15

1 Introduction

Uncertain and risky events are ubiquitous in society. While economic agents cannot eliminate all of the exogenous source of risk, they can exercise a certain control over the amount of risk they face through the market process.[1] Specifically, markets and prices allocate resources to different risky activities, and among different agents. For instance, when a firm undertakes a risky project in the real sector, the size of the project as well as the share of risk borne by each shareholder depend on market forces in both the real and financial sectors. In particular, the choice and allocation of risk depend on the prices of goods in the real sector as well as the prices of financial instruments. These prices depend, in turn, on the preferences of agents, the alternative assets of the shareholders, the market structure, and the exogenous source of risk.

Yet, in the standard framework of industrial organization, markets and prices play no role in determining which types of risk are undertaken by firms and which groups of agents bear the risk. Rather, risk vanishes under the postulate that firms maximize expected profit, even if their shareholders are risk-averse. The risk-neutrality of firms owned by risk-averse shareholders is generally justified on the grounds that the shareholders' portfolios of assets are well-diversified, to the point of eliminating any exposure to, and concern for risk.[2] In other words, while the shareholders are risk-averse, portfolio diversification induces their firms to *act* as risk-neutral, and, thus, to maximize expected profit.[3]

There are two main issues with this justification. First, the market process by which shareholders diversify their portfolio is not modeled. The

[1]The reduction of risk comes at a cost, so that, even if feasible, an economic agent would not necessarily eliminate risk all together.

[2]See Tirole (1988, pp. 34-35), including footnote 61, and Salanié (1997, p. 53).

[3]Another argument in support of the risk-neutrality of firms is that shareholders are risk-neutral. If risk-averse agents owned shares of a risky asset, they could benefit from an arbitrage opportunity as well as rid themselves of any exposure to risk by selling their shares to risk-neutral agents. Hence, all risky assets would be owned by risk-neutral shareholders. Risk-neutral shareholders would invest all of their wealth in the asset with the highest expected return, and, thus, all assets would have the same rate of return, which is inconsistent with observed behavior.

diversification of assets is costly and might not benefit all shareholders in the same way. Moreover, the interplay between the shareholders' portfolio selection and the firms' decisions is an important link. On the one hand, the interaction of the shareholders in the financial market influence the behavior of the firms in the real market, which is not necessarily one of maximizing expected profit. On the other hand, the allocation of risk through the financial market depends on the distribution of real profit, which, in turn, depends on the decisions of the firms. Second, the high variability of all market indicators makes it difficult to believe that portfolio diversification renders shareholders immune to risk. Indeed, the allocation of wealth among many assets only reduces, but cannot eliminate, the unsystematic risk that emanates from each risky asset. Moreover, systematic risk remains and affects the payoffs of all assets. Thus, despite the availability of a wide range of financial instruments, shareholders must accept risk. Recent financial events have further called into question the belief that risk can be eliminated. For instance, *The Economist* writes:[4]

> American mutual-fund assets have declined by $2.4 trillion–a fifth of their value–since the start of 2008; in Britain, the drop is more than a quarter, or almost £130 billion ($195 billion). [...] Nor has the bad news been confined to equities. This year the value of all manner of risky investments, from corporate bonds to commodities to hedge funds, has been clobbered. *The belief that diversification into "alternative assets"could prevent investors losing money in bear markets has proved false.*

It is the purpose of this paper to address explicitly the issue of risk and the mechanism by which risk-averse shareholders diversify their portfolio of assets, in the theory of the firm. In particular, we study the role of markets and prices on the type of risky activities undertaken by a firm and the allocation of profit among shareholders. From a financial point of view, this is equivalent to studying the influence of markets and prices on the choice

[4]The excerpts are from the article "Where have all your savings gone?"of December 6, 2008 on page 13 (emphasis added).

and allocation of a risky asset issued by a firm. To that end, we embed a mean-variance approach to the shareholders' portfolio selection, pioneered by Markowitz (1952) and Tobin (1958), into the theory of the firm. By establishing an explicit link between the behavior of the shareholders and the firm's, the real and financial sectors are integrated. In particular, the payoff of the risky asset depends on the level of output, and reflects the uncertainty that emanates from the real sector.

To that end, we consider a monopoly initially owned by an entrepreneur (the managing shareholder) who has the ability to issue shares of a risky asset (tied to the random profit of the monopoly). In our model, the deciding shareholder of a firm, called the entrepreneur, undertakes a risky project in the real sector and interacts with the remaining shareholder, called the investor, in the financial sector. The project is risky because the firm faces a random price in the real market. The allocation of risk among risk-averse shareholders is achieved by selling shares of a risky asset in the financial market. Shares of the risky asset define the ownership structure of the firm and represent claims to the profit derived in the real sector. While the entrepreneur allocates the profit of the firm among the shareholders, the entrepreneur retains control of the firm's decisions. Specifically, the entrepreneur decides both the level of output and the ownership structure of the firm.

We begin by studying the Nash equilibrium with a competitive financial market. We first show that there is in general no Nash equilibrium. The reason is that, if the financial price is given, the firm has an incentive to increase the total number of shares to infinity, which yields no solution for the managing shareholder. We then consider two types of restrictions. The first type of restriction is to equate the total number of shares to output so that each share is a claim to the profit of one unit of output. The second type of restriction is to set exogenously the number of shares. We show that both restrictions yield existence of a Nash equilibrium with a competitive financial market. Also, in both cases, financial access, i.e, floating part of the shares, leads to the global acceptance of more risk and, hence, to an increase of equilibrium output. However, the limits of this increase in output, as the fraction of stock floated tends to 1, are different according to the assumption made. With a fixed number of shares output equals the monopoly solution, with or without risk depending on the risk-aversion of the investor. With the number of shares equal to total output the latter approaches the perfect equilibrium solution, again with or without risk.

We then consider the Stackelberg equilibrium under two scenarios. In the first one, the entrepreneur is the leader (sophisticated agent) whereas the investor is the follower (naive agent). In the second one, we reverse roles by having the investor as the leader. These scenarios lead to the basic results stated above, but differ about the particular optimal solutions. Only the competitive equilibrium is Pareto efficient; both Stackelberg equilibrium lead to smaller output levels. In the competitive market, the fraction of shares allocated to each agent is directly proportional to their respective coefficient of risk aversion; when any of the agents assumes a leading role, that fraction is distorted to favor the leader goals. There are also differences on how the fraction of shares floated and the equilibrium output change with the risk aversion coefficients. The fraction of shares floated always increases when risk aversion of the entrepreneur also increases and decreases when risk aversion of the investor increases. In the competitive market and when the entrepreneur is leading, that fraction varies from zero to 1 when the risk aversion coefficient of the entrepreneur goes from zero to infinity (or the risk aversion coefficient of the investor goes from infinity to zero). But when the leader is the investor, the fraction of shares sold varies from zero to one half; the investor never demands more than half the shares in order to depress the financial price.

The relationship between risk and firm behavior has been present in the literature for some decades. Baron (1970), Baron (1971), Sandmo (1971), and Leland (1972) studied the impact of risk aversion on the decisions of a risk averse firm in a competitive and in an imperfectly competitive market. However, these early works made no attempt to relate behavior of the firm with its ownership structure or the functioning of the financial market. Later works have established a relationship between real and financial sectors: Dotan and Ravid (1985), Prezas (1988), Brander and Lewis (1986) and Showalter (1995), arrived there while studying the problem of optimal debt-equity allocation; Jain and Mirman (2000) work on insider trading also shows that both sectors are related. Mirman and Santugini (2013) (hereafter referred as MS) on risk-sharing and financial markets goes much further. They analyze a model with a risk-averse owner of a monopolist firm (the entrepreneur) facing the option of selling part of the stock of his firm to a risk-averse outside investor. The entrepreneur retains control over all decisions of the firm, notably on the quantity of output supplied in the real market. To optimize his utility this entrepreneur must take into account, simultaneously, his decisions on the real and on the financial market, because his final expected wealth depends on both. This dual perspective distinguishes this model from most of the previous literature and integrates real and financial equilibrium.

The paper is organized as follows. After this introduction, Section 2 studies the Nash equilibrium with a competitive financial market, whereas Section 3 considers the Stackelberg equilibrium. We provide concluding remarks in Section 4.

2 Nash Equilibrium with Competitive Financial Market

In this section, we present a general model combining the behavior of the firm (in the real and financial sectors) and the behavior of the shareholders. We then establish conditions under which there exists a Nash equilibrium with a competitive financial market. In the next section, we provide conditions under which there exists a Stackelberg equilibrium with a non-competitive financial market.

2.1 Set Up

Consider a firm that is a monopoly in a real market and has access to the financial market.[5] In the real market, the firm faces a random demand with

[5] The adjective *real* refers to the sector of goods and services other than those of financial nature.

known distribution and chooses the level of output $q \geq 0$. Specifically, the random price corresponding to supplying q units is $\tilde{p}_R = P_R(q) + \tilde{\varepsilon}$ where $P_R(q)$ is the expected inverse demand and $\tilde{\varepsilon}$ is a normally-distributed shock.[6]

Assumption 2.1. $\tilde{\varepsilon} \sim N(0, \sigma^2)$.

The random profit of the firm is thus $\pi(q, \tilde{\varepsilon}) = (P_R(q) + \tilde{\varepsilon})q$. The expected profit is assumed to be strictly concave in the level of output.

Assumption 2.2. $P''_R(q)q + 2P'_R(q) < 0$.

In the financial sector, the firm issues $S \in \Omega_S \subseteq \Re_+$ equity shares.[7] Each share is a claim of $\frac{1}{S}$ of the total profit so that each share receives a random payoff $\pi(q, \tilde{\varepsilon})/S$. In addition to choosing the total number of shares, the firm decides on the fraction $1 - \omega \in [0, 1]$ of the shares to be sold in the financial market at unit price p_F.[8] Hence, the variable ω defines the ownership structure of the firm, which specifies the allocation of the random profit among the shareholders.

The objective of each shareholder is to maximize the expected utility of final wealth. Each shareholder diversifies wealth between the risky asset issued by the firm and a risk-free asset. Without loss of generality, we assume that there are only two shareholders, i.e., an entrepreneur and an investor. The entrepreneur is the founder of the firm and the original claimant of the profit generated by his entrepreneurial prospects. The entrepreneur is also the managing shareholder of the firm, making the output decision, issuing the total number of shares, and deciding on the number of shares to be floated. Having no initial wealth, the entrepreneur's random final wealth is

$$\widetilde{W}'_E = \omega \cdot \pi(q, \tilde{\varepsilon}) + p_F \cdot (1 - \omega) \cdot S \tag{1}$$

where $\omega \cdot \pi(q, \tilde{\varepsilon})$ is the entrepreneur's portion of the random profit of the firm and $p_F \cdot (1 - \omega) \cdot S$ is the wealth generated from selling $(1 - \omega) \cdot S$ shares at

[6] The subscript R refers to the *real sector* and the tilde sign differentiates a random variable from its realization.

[7] The type of restriction imposed on the set Ω_S turns out to be key for the existence of the equilibrium and the comparative analysis.

[8] The subscript F refers to the *financial sector*.

unit price p_F, and investing $p_F \cdot (1-\omega) \cdot S$ in a risk-free asset with a rate of return normalized to one.

Unlike the entrepreneur, the investor does not have entrepreneurial prospects and has no direct control over the decisions of the firm. The investor uses his initial wealth $W_I > 0$ to purchase shares of the risky asset and the risk-free asset. Hence, the investor's random final wealth is

$$\widetilde{W_I'} = W_I + \pi(q, \tilde{\varepsilon})z/S - p_F z \tag{2}$$

where z is the number of shares purchased by the investor. Here, $W_I - p_F z$ is invested in the risk-free asset and $\pi(q, \tilde{\varepsilon})z/S$ is the random payoff corresponding to z shares of the risky asset. Note that the return on a share of the firm is $\pi(q, \tilde{\varepsilon})/S - p_F$.

Each shareholder maximizes the expected utility of final wealth defined by (1) or (2). The shareholders are assumed to be risk-averse in final wealth with constant absolute risk aversion (CARA).

Assumption 2.3. *The coefficients of absolute risk aversion are $a_E > 0$ and $a_I > 0$ for the entrepreneur and the investor, respectively.*[9]

From (1), given that $\tilde{p}_R = P_R(q) + \tilde{\varepsilon}$, the certainty equivalent of the entrepreneur is[10]

$$CE_E = \omega \cdot P_R(q)q + p_F \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2 q^2/2. \tag{3}$$

Here, $\omega \cdot P_R(q)q + p_F \cdot (1-\omega) \cdot S$ is the expected payoff to the entrepreneur from the real and financial sectors weighted by the level of ownership. The term $a_E\sigma^2\omega^2q^2/2$ is the risk premium of the entrepreneur. The risk premium plays the role of a cost, due to risk aversion, imposed on the entrepreneur for bearing part of the risk. From (2), the certainty equivalent of the investor is

$$CE_I = W_I + (P_R(q)q/S - p_F)z - a_I\sigma^2(q/S)^2 z^2/2 \tag{4}$$

[9]In other words, utility functions for final wealth x are exponential: $u(x; a) = -e^{-ax}, a \in \{a_E, a_I\}$.

[10]The expected utility of the entrepreneur is $\mathbb{E}u(\widetilde{W_E}; a_E) = -e^{-a_E CE_E}$, where $\mathbb{E}$ is the expectation operator.

where $W_I + (P_R(q)q/S - p_F)z$ is the expected mean of final wealth and $a_I\sigma^2(q/S)^2z^2/2$ is the risk premium.

2.2 Equilibrium

Having described the model, we now define the Nash equilibrium with a competitive financial market. The entrepreneur and the investor move simultaneously in a Nash equilibrium. The financial sector is perfectly competitive, i.e., the financial price is given, and, thus, neither the entrepreneur nor the investor can take into account the effect of their decisions on the financial price. In equilibrium, the price of the risky asset clears the financial market by equating the quantity demanded by the investor with the quantity supplied by the firm (or the entrepreneur). The equilibrium consists of the firms' decisions made by the entrepreneur $\{q^*, \omega^*, S^*\}$, the investor's amount of shares of the risky asset z^*, and the financial price p_F^*. The entrepreneur's decisions $\{q^*, S^*\}$ have a direct effect on the investor's payoffs. However, the investor's decision has no influence on the entrepreneur's payoffs. Both shareholders are affected indirectly by each other through the financial price.

Definition 2.4. *The tuple $\{q^*, \omega^*, S^*, z^*, p_F^*\}$ is a Nash equilibrium with a competitive financial market if*

1. *Given $\{q^*, S^*\}$ and p_F^*, the investor's quantity demanded for the risky asset is*

$$z^* = \arg\max_{z \geq 0} \left\{W_I + (P_R(q^*)q^*/S^* - p_F^*)z - a_I\sigma^2(q^*/S^*)^2z^2/2\right\}. \quad (5)$$

2. *Given p_F^*, subject to $q \geq 0, \omega \in [0,1], S \in \Omega_S$,*

$$\{q^*, \omega^*, S^*\} = \arg\max_{q,\omega,S} \left\{\omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2\right\}. \quad (6)$$

3. *Given $\{\omega^*, S^*, z^*\}$, $p_F^* > 0$ satisfies the market-clearing condition $z^* = (1-\omega^*)S^*$.*

Proposition 2.5 states that in the absence of a restriction on the set for the total number of shares, there is no equilibrium. Allowing the firm to optimize on the number of shares gives the entrepreneur the incentive to increase S to infinity. The non-existence result is due to the fact the the financial market is competitive, i.e., the financial price is taken as given.

Proposition 2.5. *Suppose that $\Omega_S = \Re_+$. Then, there exists no Nash equilibrium with a competitive financial market.*

Proof. From (6), given $p_F^* > 0$, there is no solution for S^*. $\square$

In order to obtain existence of an equilibrium, the set Ω_S must be restricted. We consider two types of restrictions. Both types of restrictions essentially reduce the number of decisions for the entrepreneur from three to two. We first provide the equilibrium values under each restriction. We then compare the two approaches.

The first is to equate the total number of shares to output so that each share is a claim to the profit of one unit of output. This assumption retains the idea that the firm wishes to increase the number of shares to increase the proceeds from the financial market. At the same time, it allows for the existence of an equilibrium. Indeed, the total number of shares cannot go to infinity because, being equal to output, it is limited by the real demand function. Proposition 2.6 characterizes the equilibrium as studied in Mirman and Santugini (2013). Note that existence is only possible when the entrepreneur faces an unsharable cost of entrepreneurship. Otherwise, there is no risk sharing.[11]

Proposition 2.6. *Suppose that $\Omega_S = \{S | S = q \in \Re_+\}$. Then, there exists a Nash equilibrium with a competitive financial market as long as there is an unsharable cost of entrepreneurship. In equilibrium, q^* satisfies*

$$\omega^* \cdot (P_R'(q^*)q^* + P_R(q^*)) + (1 - \omega^*) \cdot P_R(q^*) = \omega^* a_E \sigma^2 q^*, \tag{7}$$

[11] The presence of the unsharable cost of entrepreneurship is necessary for the Hessian matrix to be negative definite.

the allocation of risk is defined by

$$\omega^* = \frac{a_I}{a_I + a_E}, \tag{8}$$

and $S^* = q^*$. Moreover, the investor's quantity demanded is

$$z^* = \frac{P_R(q^*) - p_F^*}{a_I \sigma^2} \tag{9}$$

and the financial price is

$$p_F^* = P_R(q^*) - (1 - \omega^*) a_I \sigma^2 q^*. \tag{10}$$

Proof. See Mirman and Santugini (2013). □

The second type of restriction is to set exogenously the number of shares. Proposition 2.7 characterizes the equilibrium when the total number of shares is fixed. Hence, the firm chooses only output and ownership, i.e., $\{q, \omega\}$.[12]

Proposition 2.7. *Suppose that $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$. Then, there exists a Nash equilibrium with a competitive financial market. In equilibrium, output q^* satisfies*

$$P_R'(q^*) q^* + P_R(q^*) = \omega^* a_E \sigma^2 q^*, \tag{11}$$

the allocation of risk is defined by

$$\omega^* = \frac{a_I}{a_I + a_E}, \tag{12}$$

and $S^* = \overline{S}$. Moreover, the investor's quantity demanded is

$$z^* = \frac{P_R(q^*) q^* / \overline{S} - p_F^*}{a_I \sigma^2 (q^* / \overline{S})^2} \tag{13}$$

and the financial price is

$$p_F^* = P_R(q^*) q^* / \overline{S} - (1 - \omega^*) a_I \sigma^2 q^{*2} / \overline{S}. \tag{14}$$

[12] Alternatively, setting an exogenous upper bound on the number of shares also yields existence, i.e., $\Omega_S = [0, \overline{S}], \overline{S} \in (0, \infty)$.

Proof. Given Ω_S, $S^* = \overline{S}$. The first-order condition corresponding to (5) evaluated at $S^* = \overline{S}$ yields (13). Next, plugging (13) and $S^* = \overline{S}$ into the market-clearing equilibrium $z^* = (1 - \omega^*)S^*$ yields (14). Finally, the first-order conditions corresponding to (6) evaluated at $S = \overline{S}$ are

$$q : \omega \cdot [P'_R(q)q + P_R(q)] - \omega^2 a_E \sigma^2 q = 0, \tag{15}$$

$$\omega : P_R(q)q - p^*_F \overline{S} - a_E \sigma^2 \omega q^2 = 0, \tag{16}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Rearranging (15) yields (11). Plugging (14) into (16) and solving for ω^* yields (12). □

2.3 Discussion

Having characterized the equilibrium under two types of restrictions for Ω_S, we now use Propositions 2.6 and 2.7 to compare the equilibrium values. We begin by noting that the restriction for Ω_S has an effect on the level of output, but not on the allocation of risk. Indeed, from (8) and (12), the fraction of shares to be floated is independent of the choice of Ω_S and depends only on the relative size of the risk aversion coefficients. If $a_E/a_I \to 0$, then $\omega^* \to 1$ and the entrepreneur bears all the risk, i.e., no floating. If $a_E/a_I \to \infty$, then $\omega^* \to 0$ and the investor bears all the risk, i.e., 100% floating. Unlike the allocation of risk, the level of output does depend on the choice of Ω_S. Specifically, under $\Omega_S = \{S|S = q \in \Re_+\}$, from (7), access to a financial market induces hybrid behavior for a monopolist, that is, a convex combination of monopoly and perfect competition in the real sector. Under $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$, from (11), setting an exogenous number of shares removes this hybrid behavior. It follows that the output level is always smaller under $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$ than under $\Omega_S = \{S|S = q \in \Re_+\}$. Moreover, the equilibrium output under (11) is Pareto optimal. Indeed, substituting (12) into (11) and rearranging the left-hand side yields

$$P'_R(q^*)q^* + P_R(q^*) = a_E \sigma^2 \omega^2 q^* + a_I \sigma^2 \cdot (1 - \omega)^2 q^*. \tag{17}$$

Result (17) indicates that the total marginal revenue of output (the left-hand side of (17)) is equal to the total marginal cost of risk for *both* agents (the right-hand side (17)).[13]

Next, from (7) and (11), the restriction on Ω_S has no effect on most of the comparative analysis for output. Indeed, the direction of the effect of an increase in a_I or σ^2 is independent of the choice of Ω_S. Specifically, a more risk-averse investor induces the firm to decrease production, i.e., $\partial q^*/\partial a_I < 0$. Similarly, regardless of the choice of Ω_S, an increase in the variance of the shock increases the marginal cost of bearing some risk, i.e., it increases the right-hand side of both (7) and (11). This induces the firm to decrease output, i.e., $\partial q^*/\partial \sigma^2 < 0$.

However, the effect of the entrepreneur's risk aversion on the level of output depends on the restriction imposed on the total number of shares. Indeed, under $\Omega_S = \{S | S = q \in \Re_+\}$, from (7) and (8), the sign of $\partial q^*/\partial a_E$ is ambiguous.[14] However, under $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$, from (11) and (12), $\partial q^*/\partial a_E < 0$. When the number of shares is tied to the level of output, an increase in a_E has an effect on both sides of (7). Specifically, an increase in a_E induces the entrepreneur to sell a larger fraction of the firm, i.e., ω^* decreases. This, in turn, has an effect not only on the cost of risk (the right-hand side of (7)), but also on the firm's ability to exercise market power (the left-hand side of (7)).

To understand this difference in the comparative analysis, consider now the variance of profit (which is linked to the risk premium paid to the investor). Letting $\mathbb{V}$ be the variance operator, $\mathbb{V}\pi_R(q^*, \tilde{\varepsilon}) = \sigma^2 q^{*2}$ reflects the degree to which the entrepreneur takes risk on behalf of the firm, which is different from the risk borne by the entrepreneur. Note that the effect of risk aversion on risk-taking depends on the restriction imposed on the set Ω_S through the level of output. As discussed, under $\Omega_S = \{S | S = q \in \Re_+\}$, it is possible for $\partial q^*/\partial a_E > 0$, which implies that $\partial \mathbb{V}\pi_R(q^*, \tilde{\varepsilon})/\partial a_E > 0$ if and only if $-P_R'(q^*) > a_I \sigma^2$.

Although risk-averse shareholders have an aversion for risk, their rewards

[13]See Appendix A.

[14]Specifically, $\partial q^*/\partial a_E > 0$ if and only if $-P_R'(q^*) > a_I \sigma^2$.

(expected return) depend positively on the amount of risk the firm takes. In other words, the higher the risk premium of an investor, the higher the premium (in terms of expected returns) given to a shareholder to bear part of the risk of the firm. This conflict between shareholders disdain for risk and the increase in the payment when risk increases is important. Under $\Omega_S = \{S|S = q \in \Re_+\}$, the entrepreneur increases output in order to increase the expected payment for risk sharing and thus to induce the investor to take on more risk. This is only possible if the entrepreneur makes the firm riskier, i.e., offers a higher risk premium. A more risk-averse entrepreneur makes the firm's variance greater in order to increase risk sharing by increasing the risk premium corresponding to each share sold to the investors. For $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$, the entrepreneur decreases output so that the firm takes on less risk, i.e., $\partial q^*/\partial a_E < 0$ implies that $\partial \mathbb{V}\pi_R(q^*, \tilde{\varepsilon})/\partial a_E < 0$. That is, with a fixed total number of shares, the entrepreneur only reacts to an increase in his risk aversion but has no concern for encouraging the investor to take on more risk.

We conclude this discussion by comparing the limiting cases. We begin with the restriction $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$. As $\sigma^2 \to 0$, the level of output equals the solution for a risk-averse monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_I = 0$, $\omega^* = 0$ and the level of output tends to the solution for a monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_I \to \infty$, $\omega^* \to 1$ and the level of output tends to the solution of a monopoly owned solely by a risk-averse entrepreneur, i.e., $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$. This is shown in Figure 1.[15]

As $a_E = 0$, $\omega^* = 1$ and the level of output equals the solution for a risk-averse monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_E \to \infty$, $\omega^* \to 0$ the level of output tends to the solution of a risk-averse monopoly owned by the investor who takes on all the risk, i.e., $P_R(q^*) = a_I\sigma^2 q^*$. See Figure 2.

The limiting cases for $\Omega_S = \{S|S = q \in \Re_+\}$ are as follows. As $\sigma^2 \to 0$, the level of output is a linear combination of the solution for a risk-averse monopoly facing no risk and for a competitive firm also facing no risk, i.e.,

[15] In the following graphs we assumed, for simplicity, that real demand is linear.

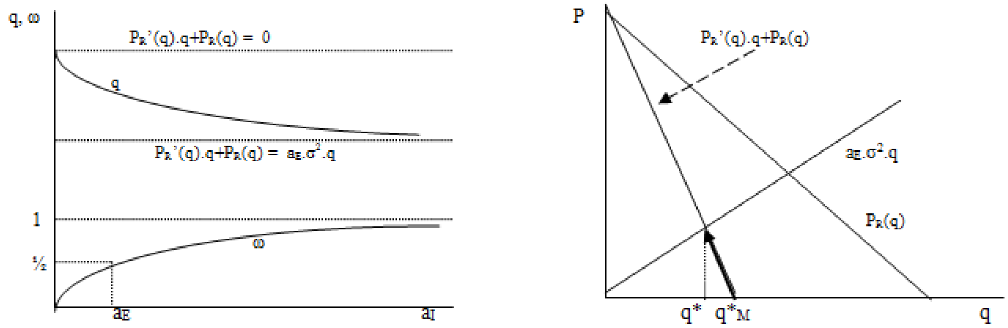

Figure 1: The Effect of a_I on q^* and ω^* when $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$.

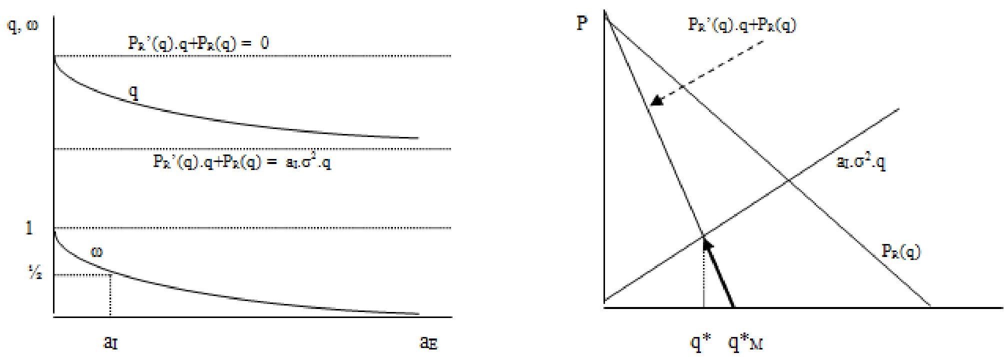

Figure 2: The Effect of a_E on q^* and ω^* when $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$

$\omega^* \cdot (P_R'(q^*)q^* + P_R(q^*)) + (1-\omega^*) \cdot P_R(q^*) = 0$. As $a_I = 0$, $\omega^* = 0$ and the level of output equals the solution for a risk-averse perfectly competitive firm facing no risk, i.e., $P_R(q^*) = 0$. As $a_I \to \infty$, $\omega^* \to 1$ and the level of output tends to the solution of a risk-averse monopoly facing risk who does not share risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$. See Figure 3.

Finally, as $a_E = 0$, $\omega^* = 1$ and the level of output equals the solution for a risk-averse monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_E \to \infty$, $\omega^* \to 0$ the level of output tends to the solution of a risk-averse competitive firm owned by the investor who takes on all the risk, i.e., $P_R(q^*) = a_I\sigma^2 q^*$.[16]. See Figure 4.

[16]This is the case in which output may go up or down, depending on which of the extreme solutions is larger

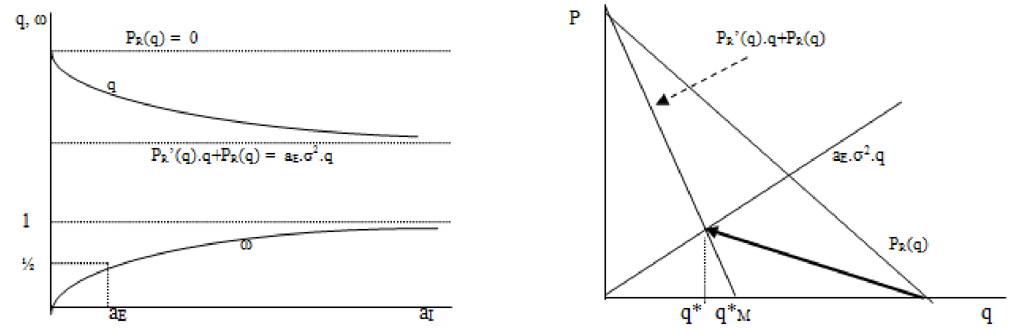

Figure 3: The Effect of a_I on q^* and ω^* when $\Omega_S = \{S|S = q \in \Re_+\}$

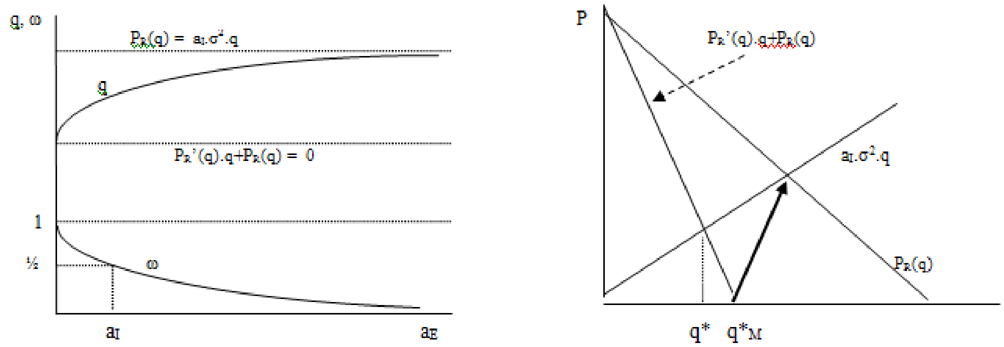

Figure 4: The Effect of a_E on q^* and ω^* when $\Omega_S = \{S|S = q \in \Re_+\}$

3 Stackelberg Equilibrium with Non-Competitive Financial Market

In this section, we consider the Stackelberg equilibrium in which one agent is sophisticated whereas the other agent is naive. In other words, one of the agents is a leader and the other one is a follower. The Stackelberg environment is not compatible with a competitive financial market. Hence, the leader (whether the entrepreneur or the investor) has market power in the financial sector. We begin by showing that there exists a Stackelberg equilibrium with a leading entrepreneur without the need to impose any restrictions on the Ω_S. However, the total number of shares and the financial price remain undetermined. i.e., one depends on the arbitrary choice of the other. We then show that there is no Stackelberg equilibrium with a leading investor unless restrictions are imposed on the Ω_S.

3.1 Leading Entrepreneur

We first define the Stackelberg equilibrium with the entrepreneur as the leader. We then provide equilibrium values.

Definition 3.1. *The tuple $\{q^*, \omega^*, S^*, z^*(q^*, \omega^*, S^*), p_F^*\}$ is a Stackelberg equilibrium (leading entrepreneur) with a non-competitive financial market if*

1. *Given $\{q^*, \omega^*, S^*\}$ and p_F^*, the investor's quantity demanded for the risky asset is*

$$z^*(q^*, \omega^*, S^*) = \arg\max_{z \geq 0} \left\{ W_I + (P_R(q^*)q^*/S^* - p_F^*)z - a_I\sigma^2(q^*/S^*)^2 z^2/2 \right\}. \tag{18}$$

2. *Given $z^*(q, \omega, S)$, subject to $q \geq 0, \omega \in [0,1], S \in \Omega_S$,*

$$\{q^*, \omega^*, S^*\} = \arg\max_{q,\omega,S} \left\{ \omega P_R(q)q + D^*(q, \omega, S) \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2 q^2/2 \right\} \tag{19}$$

   where $p_F = D^*(q, \omega, S)$ is the inverse financial demand defined by $z^*(q, \omega, S) = (1-\omega)S$.

3. *Given $\{q^*, \omega^*, S^*, z^*(q^*, \omega^*, S^*)\}$, $p_F^* > 0$ satisfies the market-clearing condition $z^*(q^*, \omega^*, S^*) = (1-\omega^*)S^*$.*

Proposition 3.2 states that there exists an equilibrium when the entrepreneur is the leader. Hence, another way to reestablish existence without any restriction on the total number of shares is to assume that the entrepreneur has market power in the financial sector. However, the total number of shares and the financial price cannot be uniquely and independently determined. The reason is that the inverse financial demand is inversely proportional to S so that, from (19), the total number of shares has no effect on the entrepreneur's certainty equivalent.

Proposition 3.2. *Suppose that $\Omega_S = \Re_+$. Then, there exists a Stackelberg equilibrium with a sophisticated entrepreneur. In equilibrium, output q^* satisfies*

$$P_R'(q^*)q^* + P_R(q^*) = \omega^* a_E \sigma^2 q^*, \tag{20}$$

the allocation of risk is defined by

$$\omega^* = \frac{2a_I}{2a_I + a_E}. \tag{21}$$

Moreover, the investor's quantity demanded is

$$z^*(q^*, \omega^*, S^*) = \frac{P_R(q^*)q^*/S^* - p_F^*}{a_I\sigma^2(q^*/S^*)^2}. \tag{22}$$

and

$$S^* p_F^* = P_R(q^*)q^* - (1-\omega^*)a_I\sigma^2 q^{*2} \tag{23}$$

where the total number of shares and the financial price cannot be determined separately.

Proof. The first-order condition corresponding to (18) yields (22). Next, plugging (22) (for any q, ω, and S) into $z^*(q, \omega, S) = (1-\omega)S$ and solving for the inverse financial demand function yields

$$D^*(q, \omega, S) = P_R(q)q/S - (1-\omega)a_I\sigma^2 q^2/S. \tag{24}$$

Plugging (24) into (19) yields the entrepreneur's maximization problem

$$\max_{q,\omega} \left\{ P_R(q)q - (1-\omega)^2 a_I \sigma^2 q^2 - a_E\sigma^2\omega^2 q^2/2 \right\} \tag{25}$$

where S has no effect on the entrepreneur's certainty equivalent. From (23), it follows that

$$p_F^* = \left(P_R(q^*)(q^*) - a_I\sigma^2\omega q^{*2}\right)/S^* \tag{26}$$

S^* and thus p_F^* are undefined. The first-order condition corresponding to (25) are

$$q : P_R'(q)q + P_R(q) - 2(1-\omega)^2 a_I \sigma^2 q - a_E \sigma^2 \omega^2 q = 0, \tag{27}$$

$$\omega : 2(1-\omega) a_I \sigma^2 q^2 - a_E \sigma^2 \omega q^2 = 0, \tag{28}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Solving (28) for ω^* yields (21). Plugging (21) into (27) and rearranging yields (20). Plugging q^* and ω^* into (24) and multiplying by S^* yields (23). □

Having characterized the Stackelberg equilibrium, we now compare equilibrium values under Nash and Stackelberg. First, comparing (8) or (12) with (21), it follows that the entrepreneur shares less risk under Stackelberg than under Nash (regardless of the restriction on Ω_S for Nash). This is due to the fact that the investor's coefficient of risk aversion is weighed twice under Stackelberg. Under Stackelberg, the entrepreneur takes into account the effect of an increase in shares offered in the financial price through the marginal risk cost of the investor. Hence, the entrepreneur sets a smaller float of shares in order to increase the financial price. The firm's output is thus lower under Stackelberg than under Nash because the right-hand side in (20) is now bigger than in (7) or (11)[17]. The signs of the effects of the risk coefficients on ω^* and q^*, as well as the limits of ω^* (when the risk coefficients approach zero or infinity) remain unchanged. The limits of q^* when a_E or a_I tend to zero or when a_I tends to infinity are also left unchanged. However, under Stackelberg, when a_E tends to infinity, $\omega^* \to 0$ and output q^* satisfies $P_R'(q^*)q^* + P_R(q^*) = 2a_I\sigma^2 q^*$ in the limit.

3.2 Leading Investor

Having considered the case of a leading entrepreneur, we now study the Stackelberg equilibrium with a leading investor.

Definition 3.3. *The tuple $\{q^*(z^*), \omega^*(z^*), S^*(z^*), z^*, p_F^*\}$ is a Stackelberg equilibrium (leading investor) with a non-competitive financial market if*

[17] From expression (20), it can be shown that the equilibrium output is no longer Pareto optimal.

1. *Given $\{q^*(z), \omega^*(z), S^*(z)\}$, the investor's quantity demanded for the risky asset is*

$$z^* = \arg\max_{z\geq 0} \left\{W_I + (P_R(q^*(z))q^*(z)/S^*(z) - D^*(z))z - a_I\sigma^2(z/S^*(z))^2/2\right\} \tag{29}$$

where $p_F = D^*(z)$ is the inverse financial demand defined by $z = (1 - \omega(z))S(z)$.

2. *Given p_F^*, subject to $q \geq 0, \omega \in [0,1], S \in \Omega_S$,*

$$\{q^*(z), \omega^*(z), S^*(z)\} = \arg\max_{q,\omega,S} \left\{\omega P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2\right\} \tag{30}$$

3. *Given $\{q^*(z^*), \omega^*(z^*), S^*(z^*), z^*\}$, $p_F^* > 0$ satisfies the market-clearing condition $z^* = (1 - \omega^*(z^*))S^*(z)$.*

Proposition 3.4. *Suppose that $\Omega_S = \Re_+$. Then, there exists no Stackelberg equilibrium with a leading investor.*

Proof. From (30), given $p_F^* > 0$, there is no solution for S^*. $\square$

In order to obtain an equilibrium for 3.3 with a leading investor, we must guaranty there is a solution for the entrepreneur's optimization problem. As we have seen before, this requires some kind of constraint on S. For instance, Proposition 3.5 provides the equilibrium values under Stackelberg when the total number of shares is set exogenously. In equilibrium, the decisions of the entrepreneur do not depend on z directly. Hence, notation is simplified by writing $\{q^*, \omega^*, S^*\}$.

Proposition 3.5. *Suppose that $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$. Then, there exists a Stackelberg equilibrium with a leading investor. In equilibrium, output q^* satisfies*

$$P_R'(q^*)q^* + P_R(q^*) = \omega^* a_E\sigma^2 q^*, \tag{31}$$

the allocation of risk is defined by

$$\omega^* = \frac{a_I + a_E}{a_I + 2a_E}, \tag{32}$$

and $S^* = \overline{S}$. Moreover, the investor's quantity demanded is

$$z^* = \frac{a_E \overline{S}}{2a_E + a_I}, \tag{33}$$

and the financial price is

$$p_F^* = P_R(q^*)q^*/\overline{S} - \frac{a_E + a_I}{2a_E + a_I} a_E \sigma^2 q^{*2}/\overline{S}. \tag{34}$$

Proof. Given Ω_S, $S^*(z) = \overline{S}$. The first-order conditions corresponding to (30) are

$$q : \omega \cdot [P_R'(q)q + P_R(q)] - a_E \sigma^2 \omega^2 q = 0, \tag{35}$$

$$\omega : P_R(q)q - p_F^* \overline{S} - a_E \sigma^2 \omega q^2 = 0, \tag{36}$$

evaluated at $q = q^*(z)$ and $\omega = \omega^*(z)$. Solving (36) yields

$$\omega^* = \frac{P_R(q^*(z))q^*(z) - p_F^* \overline{S}}{a_E \sigma^2 q^*(z)^2}, \tag{37}$$

which does not depend on z directly. Next, plugging $S^*(z) = \overline{S}$ and (37) into the market-clearing condition $z = (1 - \omega^*(z))S^*(z)$ and solving for the inverse financial demand yields

$$D^*(z) = P_R(q^*(z))q^*(z)/\overline{S} - \left(1 - \frac{z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2/\overline{S}. \tag{38}$$

Plugging (38) into the investor's maximization problem yields

$$\max_z \left\{ W_I + \left(1 - \frac{z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2 z/\overline{S} - a_I \sigma^2 q^*(z)^2 z^2/(2\overline{S}^2) \right\} \tag{39}$$

where, from (35), $q^*(z)$ does not depend on z. The first-order condition is

$$\left(1 - \frac{2z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2/\overline{S} - a_I \sigma^2 q^*(z)^2 z/\overline{S}^2 = 0 \tag{40}$$

evaluated at $z = z^*$ yielding (33). Next, plugging (33) into (38) yields (34).

Plugging (34) into (37) yields (32). □

As in Nash and Stackelberg with a leading entrepreneur, the allocation of risk under Stackelberg with a leading investor depends on the risk-aversion coefficients. However, in Stackelberg with a leading investor, it is the investor who has to take into account the effect of an increase in shares demanded in the financial price through the marginal risk cost of the entrepreneur. Hence, the fraction of shares sold under Stackelberg with a leading investor is less than under Nash, regardless of the restriction imposed on Ω_S.

Under Stackelberg, the fraction of shares sold can be smaller or bigger, depending on the relative size of a_I and a_E. Since (32) is always larger than (8) or (12), the right-hand side of (31) is also larger and so the equilibrium output under Stackelberg with a leading investor is smaller[18].

The signs of the effects of the risk coefficients on ω and q remain unchanged. The value of $\omega^* = \frac{a_I + a_E}{a_I + 2a_E}$ is 1 when a_E tends to zero; but when a_I tends to zero ω equals 1/2. This is quite interesting, because now, even when the investor is risk neutral, he does not buy all the shares. Indeed, acting as a leader, he prefers to buy less than half the shares to force the entrepreneur into selling with a greater discount. When a_I approaches infinity, ω tends to 1. However, when a_E approaches infinity, ω tends 1/2, essentially for the same reason.

A corresponding behavior can be inferred about q^*. When a_E tends to zero or when a_I approaches infinity, ω^* goes to 1 and q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$. When a_I tends to zero, ω^* goes to 1/2 and q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*/2$. However, when a_E approaches infinity, the problem becomes more complicated: ω goes to 1/2, but as q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$, the right-hand side of the first-order condition goes to infinity, forcing q^* to tend to zero. This happens because, as the entrepreneur cannot sell all the shares, he must always support some of the risk; when his risk aversion increases, the only way to compensate is to decrease the output towards zero.

[18] From expression (31), it can be shown that the equilibrium output is no longer Pareto optimal.

4 Final Remarks

In this paper, we have discussed the issue of existence of an equilibrium when integrating the real and financial markets. After showing that there is in general no Nash equilibrium with a competitive financial market, we impose several restrictions on the set for the total number of shares issued. Each restrictions yields existence of an equilibrium. Another way to ensure existence is to do away with the hypothesis of competitive financial market by assuming instead that either the entrepreneur or the investor can influence the financial price.

It is important to continue studying the interaction of shareholders in markets and their influence on the behavior of the firm. The interaction between real and financial markets deserves further researching, namely introducing asymmetric information on some of the parameters, a multi-period time horizon and the possibility of learning and experimenting.

A Pareto Optimality

To see Pareto optimality, notice that (17) is the solution of

$$\max_{\omega,q} CE_E = \max_{\omega,q} \left\{\omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2\right\} \tag{41}$$

where $S = \overline{S}$, and subject to $W^* = W_I + (P_R(q)q/S - p_F^*)z - a_I\sigma^2(q/S)^2z^2/2$, for $W^* > 0$. Hence, the Lagrangian is

$$\begin{aligned}\mathcal{L} = {} & \omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2 \\ & + \lambda\left(W^* - W_I - (P_R(q^*)q/S + p_F^*)z + a_I\sigma^2(q/S)^2z^2/2\right),\end{aligned} \tag{42}$$

so that

$$\frac{\partial\mathcal{L}}{\partial q} = \omega\cdot[P_R'(q^*)q^* + P_R(q^*)] - a_E\sigma^2\omega^2q + \lambda\left[(-P_R'(q^*)q^* - P_R(q^*))z/\overline{S} + a_I\sigma^2(z/\overline{S})^2q\right] = 0. \tag{43}$$

Setting W^* so that $\lambda = -1$, and using the market-clearing condition $z = (1-\omega)\overline{S}$ or $z/\overline{S} = 1-\omega$ into (43) yields (17).

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