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Investment in a Growth Model of Non-Excludable Aggregate Capital

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Abstract:
We study the effect of investment on the dynamics of aggregate capital when different sectors of the economy compete strategically for the utilization of non-excludable capital to produce both consumption and investment goods. We consider two types of investment goods: complements and substitutes. For each case, we derive the equilibrium and provide the corresponding stationary distribution. We then compare the equilibrium with the social planner's optimal solution.

Keywords: Capital accumulation, Dynamic game, Growth, Investment, Non-excludable capital

JEL Classification: C72, C73, D81, D92, O40

1 Introduction

Capital theory is an essential aspect of economics since it conveys the importance of dynamics on the structure of the economy. It is the notion of aggregate capital in which capital is fungible that is the basis of growth theory as studied in Solow (1956) for the positive non-stochastic case, Mirman (1972, 1973) for the positive stochastic case, Cass (1965) and Koopmans (1965) for the optimal non-stochastic case, and Brock and Mirman (1972) and Mirman and Zilcha (1975) for the optimal stochastic case. Although it is important to understand the optimal path of aggregate capital, the effect of several sectors in the economy competing for the utilization of capital should also be understood. Indeed, non-excludable capital structures such as airports, harbors, roads, pipe lines, transmission grids, railroads, telecommunications lines, energy are ubiquitous. In utilizing non-excludable capital, each sector of the economy needs to consider the interests of their competitors. This is an important issue in macroeconomics because it has an effect on the accumulation of capital and economic growth.

These strategic interactions for the use of non-excludable capital give rise to different sorts of externalities. The first one to be studied in a dynamic context was the *dynamic* externality (Mirman, 1979; Levhari and Mirman, 1980), i.e., the utilization of non-excludable capital by one sector has an effect on the other sectors' payoffs. Indeed, a sector which increases its usage of telecommunication lines reduces the effective use of this capital structure by the other sectors due to fact that too few lines are created. The dynamic externality yields a greater utilization of the capital, and, therefore, a smaller steady state of capital. In a more recent paper, Koulovatianos and Mirman (2007) studies the link between market structure and industry dynamics. For instance, the public and private sectors utilize capital to produce similar final goods such as health goods. The interaction of entities in the market for final goods gives rise to a *market* externality. Koulovatianos and Mirman (2007) shows that the combination of market and dynamic externalities has an ambiguous effect on the overall utilization of the capital as well as the steady state.

All these studies consider implicitly the utilization of non-excludable capital for the production of consumption goods. Yet, various sectors of the economy also produce investment goods, engage in R&D and technological progress, which has a profound effect on the dynamics of the stock of non-excludable capital. The interaction of sectors for investing in capital gives rise to an *investment* externality, i.e., the utilization of capital from one sector in order to invest in future capital has an effect on the other sector's payoff through the appreciation of the future stock. For instance, if the productive activities of one sector improves the effectiveness of telecommunication lines, then all sectors benefit from it via a *better* stock of capital.

It is the purpose of this paper to study the dynamics of capital in a situation in which different sectors of the economy compete strategically in the utilization of capital to produce both consumption and investment goods. To that end, we adapt the Levhari and Mirman (1980) framework to gain insight into the effect of strategic investment on behavior and the dynamics of capital. In our model, sectors not only utilize a stock for the production of consumption goods, but also for the production of investment goods. The evolution of capital depends on utilization, several random shocks, as well as the investment goods, which gives rise to an investment externality.

We consider two types of investment goods: complements and substitutes. For each case, we derive the dynamic Cournot-Nash equilibrium under finite and infinite horizons. We also provide the stationary distribution corresponding to the infinite-horizon equilibrium. We then compare the outcome of the game with the social planner's optimal solution. We show that there is a tragedy of the commons in the sense that the game (compared to social planning) yields more utilization. In addition, the game leads to an increase in the production of consumption goods and a decrease in the production of investment goods. As a result, the investment externality has a negative effect on the stationary distribution of capital.

The framework we adopt is not only useful for the study of economic growth in a macroeconomy with aggregate non-excludable capital being utilized by several sectors. But it is also useful to study at the industry level in which firms use industry-specific capital to produce both consumption and investment goods. Similarly, in the study of natural resources, exploiters extract non-excludable capital goods such as stocks of fish, water, oil, which is then used for production of consumption and investment goods.

The paper is organized as follows. Section 2 presents the model and defines the equilibrium. Section 3 characterizes the equilibrium under for both complements and substitutes and provides the stationary distributions under a game. Section 4 studies the effect of the investment externality by comparing the equilibrium outcomes with the optimal solution of the social planner. Section 5 offers some concluding remarks.

2 Model and Equilibrium

In this section, we present a dynamic game in which two sectors compete in the utilization of a non-excludable capital in order to produce consumption and investment goods. Consumption goods yield immediate payoff whereas investment goods have an effect on future payoffs through the dynamics of the capital. We first present the general model. We then define the recursive Cournot-Nash equilibrium. In the subsequent sections, we characterize the equilibrium under both complementary and substitutionary investment goods. We then compare the equilibrium outcomes with the social planner's solution.

Let y_t be the stock of *efficiency* units of non-excludable capital available at the beginning of period t. Absent utilization and investment, the stock of units of capital evolves stochastically according to the rule[1]

$$\tilde{y}_{t+1} = f(y_t, \tilde{\alpha}_t) \tag{1}$$

where $f(\cdot)$ is the transition function and $\tilde{\alpha}_t$ is an iid random technological shock in period t, i.e., the shock is realized in period $t+1$.

In period t, for $j = 1, 2$, sector j utilizes $e_{j,t}$ units of capital in order to produce $c_{j,t}$ units of consumption goods and $i_{j,t}$ units of investment goods. Production is linear so that $e_{j,t} = c_{j,t} + i_{j,t}$. The production of consumption

[1] A tilde sign distinguishes a random variable from its realization.

goods yields immediate payoffs $\pi(c_{j,t})$. The sectors' utilization of the capital and their production of investment goods have an effect on the future stock. Using (1),

$$\tilde{y}_{t+1} = g(i_{1,t}, i_{2,t}, \tilde{\boldsymbol{\eta}}_t) \cdot f(y_t - e_{1,t} - e_{2,t}, \tilde{\alpha}_t) \tag{2}$$

where $g(\cdot)$ is the investment function and $\tilde{\boldsymbol{\eta}}_t$ is a N-vector of iid shocks in period t. To simplify notation, the t-subscript for indexing time is hereafter removed and the hat sign is used to indicate the value of a variable in the subsequent period, i.e., y is stock today and, given any realizations of $\boldsymbol{\eta}$ and α,

$$\hat{y} = g(i_1, i_2, \boldsymbol{\eta}) \cdot f(y - e_1 - e_2, \alpha) \tag{3}$$

is stock tomorrow. From (3), investment is needed to maintain capital and ensure its future use.

To distinguish among different horizons of the dynamic game, we use the index $\tau = 0, 1, \ldots, T$. Given a horizon and the present stock of the non-excludable capital, sector j maximizes the expected sum of discounted payoffs over utilization and production of both consumption and investment goods. Formally, for $j, k = 1, 2, j \neq k$, the τ-period-horizon value function of sector j is

$$v_j^{\tau}(y) = \max_{e_j, i_j} \{\pi(e_j - i_j) + \delta \mathbb{E} v_j^{\tau-1}(g(i_j, i_k, \tilde{\boldsymbol{\eta}}) \cdot f(y - e_j - e_k, \tilde{\alpha}))\} \tag{4}$$

where $c_j = e_j - i_j$ and $\mathbb{E}$ is the expectation operator for $\tilde{\boldsymbol{\eta}}$ and $\tilde{\alpha}$. From (4), sector k's choices have an effect on sector j's expected sum of discounted payoffs through the dynamics of capital.

In general, in a dynamic game, the value function defined in (4) might not be concave (Mirman, 1979). In addition, our model includes two inherently dynamic decisions for each sector as well as several random shocks. In order to characterize the equilibrium and study its properties under different cases of investment goods, we resort to a modified version of the Levhari and Mirman (1980) framework. The following assumptions hold for the remainder of the paper. We leave the investment function unspecified for the moment and consider several types of investment goods in the next sections.

Assumption 2.1. *The joint p.d.f. of* $\tilde{\boldsymbol{\eta}}$ *and* $\tilde{\alpha}$ *is* $\phi(\boldsymbol{\eta}, \alpha), \boldsymbol{\eta} \in (0,1)^N, \alpha \in (0,1)$. *Let* $\overline{\boldsymbol{\eta}} \equiv \mathbb{E}\tilde{\boldsymbol{\eta}}$ *and* $\overline{\alpha} \equiv \mathbb{E}\tilde{\alpha}$ *be the means of the random shocks.*

Assumption 2.2. *For* $j = 1, 2$, $\pi(c_j) = \ln c_j$.

Assumption 2.3. *For* $\alpha \in (0,1)$, $f(y - e_j - e_k, \alpha) = (y - e_j - e_k)^\alpha$.

We now define the recursive Cournot-Nash equilibrium for a T-period-horizon game (Levhari and Mirman, 1980). The equilibrium consists of the strategies of the two sectors for every horizon from the first period (when there are T periods left) to the last period (when there is no horizon). Without loss of generality, we assume that in the last period the two sectors split the stock equally and do not invest. The assumption of a log utility function implies that the allocation of the stock in the last period has no effect on the dynamic game. Condition 1 states the behavior in the last period, i.e., when the horizon is $\tau = 0$. Condition 2 states the recursive equilibrium for every horizon of the game. Expression (6) for $\tau = 1$ is consistent with statement 1, i.e., for all j, $V_j^0(y) = \ln(E_j^0(y) - I_j^0(y))$. Expression (6) for $\tau = 2, \ldots, T-1$ reflects the recursive nature of the equilibrium in which the equilibrium continuation value function for a τ-period horizon depends on the equilibrium strategies for τ'-period horizons, $\tau > \tau' \geq 0$.

Definition 2.4. *The tuple* $\{E_1^\tau(y), I_1^\tau(y), E_2^\tau(y), I_2^\tau(y)\}_{\tau=0}^T$ *is a recursive Cournot-Nash equilibrium for a* T*-period-horizon game if, for all* $y > 0$,

1. *For* $\tau = 0$, $E_1^0(y) = E_2^0(y) = y/2, I_1^0(y) = I_2^0(y) = 0$.

2. *For* $\tau = 1, 2, \ldots, T$, *for* $j, k = 1, 2, j \neq k$, *given* $\{E_k^\tau(y), I_k^\tau(y)\}$ *and* $\{E_1^t(y), I_1^t(y), E_2^t(y), I_2^t(y)\}_{t=0}^{\tau-1}$,

$$\{E_j^\tau(y), I_j^\tau(y)\} = \arg\max_{e_j, i_j} \Bigg\{ \ln(e_j - i_j) + \delta \int V_j^{\tau-1}(g(i_j, I_k^\tau(y), \boldsymbol{\eta}) \cdot (y - e_j - E_k^\tau(y))^\alpha) \cdot \phi(\boldsymbol{\eta}, \alpha) \mathrm{d}\boldsymbol{\eta} \mathrm{d}\alpha \Bigg\} \quad (5)$$

where, for any $x > 0$,

$$V_j^{\tau-1}(x) = \begin{cases} \ln(x/2), & \tau = 1 \\ \ln(E_j^{\tau-1}(x) - I_j^{\tau-1}(x)) + \delta \int V_j^{\tau-2}(\Lambda) \cdot \phi(\boldsymbol{\eta}, \alpha) \mathrm{d}\boldsymbol{\eta} \mathrm{d}\alpha, & \tau = 2, 3, ..., T \end{cases} \tag{6}$$

with

$$\Lambda \equiv g(I_1^{\tau-1}(x), I_2^{\tau-1}(x), \boldsymbol{\eta}) \cdot \left(x - \sum\nolimits_{s=1}^{2} E_s^{\tau-1}(x)\right)^{\alpha}. \tag{7}$$

3 Equilibrium Characterization

In this section, we fully characterize the equilibrium for any finite horizon. We then show that the limit of the finite-horizon equilibrium exists. In other words, there exists an equilibrium for the infinite horizon that is consistent with the sequence of finite-horizon equilibrium. Hence, the existence and the properties of the equilibrium are robust to any finite and infinite horizon. We then provide the stationary distribution for capital corresponding to the limiting case.

We begin with the case of complementary investment goods. We then repeat the analysis for the case of substitutionary investment goods. The main difference between complements and substitutes concern uniqueness. When investment goods are complementary, the equilibrium is unique whereas there is a continuum of equilibrium points with substitutionary investment goods. However, regardless of the type of investment goods, the stationary distribution of capital is unique. In the next section, we compare the equilibrium with the solution of the social planner.

3.1 Complementary Investment Goods

When investment goods are complementary, the investment function is specified as

$$g(i_1, i_2, \boldsymbol{\eta}) = i_1^{\eta_1} i_2^{\eta_2}, \tag{8}$$

$\boldsymbol{\eta} \equiv [\eta_1, \eta_2]$. Using (8), (3) is rewritten as

$$\hat{y} = i_1^{\eta_1} i_2^{\eta_2} (y - e_1 - e_2)^{\alpha}. \tag{9}$$

The investment term $i_1^{\eta_1} i_2^{\eta_2}$ reflects the complementarity of the sectors' investments. The shocks η_1 and η_2 measure the individual contribution of the investment goods toward the future stock.

Proposition 3.1 provides the utilization level as well as the production levels for consumption and investment goods corresponding to the unique equilibrium for any finite horizon. The equilibrium displays certainty equivalence, i.e., the means of the shocks are the only moments of the distribution to have an effect on decisions. Moreover, the equilibrium is in general asymmetric unless the means of the investment shocks are identical.

Proposition 3.1. *Suppose that the investment goods are complementary. Then, there exists a unique recursive Cournot-Nash equilibrium for a T-period game, $T = 1, 2, \ldots$ In equilibrium, for $\tau = 0, 1, \ldots, T$, for $j = 1, 2$, sector j utilizes*

$$E_j^{\tau}(y) = \frac{1 + \delta\overline{\eta}_j \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)}{2 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)} y \tag{10}$$

units of capital for the production of

$$C_j^{\tau}(y) = \frac{1}{2 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)} y \tag{11}$$

units of consumption goods and

$$I_j^{\tau}(y) = \frac{\delta\overline{\eta}_j \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)}{2 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}\right)^t \right)} y \tag{12}$$

units of investment goods.

Proof. We first derive utilization, investment, and value functions in the one-period horizon. We then consider a τ-period horizon and solve for utilization, investment and value functions recursively. We finally impose the initial condition given by the one-period-horizon solution.

1. Consider first the one-period horizon. Using (5), (6), and (9), for $j, k = 1, 2, j \neq k$, given $\{E_k^1(y), I_k^1(y)\}$, sector j's one-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^1(y), I_j^1(y)\} = &\arg\max_{e_j, i_j} \Big\{ \ln(e_j - i_j) + \delta\overline{\eta}_j \ln i_j + \delta\overline{\eta}_k \ln I_k^1(y) \\ &+ \delta\overline{\alpha} \ln(y - e_j - E_k^1(y)) - \delta \ln 2 \Big\}. \end{aligned} \tag{13}$$

   The first-order conditions corresponding to (13) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}}{y - e_j - E_k^1(y)}, \tag{14}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}_j}{i_j}, \tag{15}$$

   evaluated at $e_j = E_j^1(y)$ and $i_j = I_j^1(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. For $j, k = 1, 2, j \neq k$, solving (14) and (15) for the equilibrium yields the unique solution for one-period-horizon utilization and investment,

$$E_j^1(y) = \frac{1 + \delta\overline{\eta}_j}{2 + \delta\left(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}\right)} y, \tag{16}$$

$$I_j^1(y) = \frac{\delta\overline{\eta}_j}{2 + \delta\left(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}\right)} y. \tag{17}$$

   Plugging (16) and (17) for the two sectors into the objective function in (13) yields

$$V_j^1(y) = (1 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})) \ln y + \Psi_1, \tag{18}$$

   where Ψ_1 is a constant for the one-period horizon that has no effect on the solution.

2. Having solved for the one-period horizon, we consider next a τ-period horizon for which the continuation value function is of the form $V_j^{\tau-1}(y) = \kappa_{\tau-1}\ln y + \Psi_{\tau-1}$ where $\kappa_{\tau-1}$ and $\Psi_{\tau-1}$ are constants. For $j,k=1,2, j\neq k$, given $V_j^{\tau-1}(y) = \kappa_{\tau-1}\ln y + \Psi_{\tau-1}$ and $\{E_k^\tau(y), I_k^\tau(y)\}$, sector j's τ-period-horizon optimal policies satisfy

$$\{E_j^\tau(y), I_j^\tau(y)\} = \arg\max_{e_j,i_j} \{\ln(e_j - i_j) + \delta\overline{\eta}_j\kappa_{\tau-1}\ln i_j + \delta\overline{\eta}_k\kappa_{\tau-1}\ln I_k^\tau(y) + \delta\overline{\alpha}\kappa_{\tau-1}\ln(y - e_j - E_k^\tau(y)) + \delta\Psi_{\tau-1}\}. \tag{19}$$

The first-order conditions corresponding to (19) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}\kappa_{\tau-1}}{y - e_j - E_k^\tau(y)} \tag{20}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}_j\kappa_{\tau-1}}{i_j} \tag{21}$$

evaluated at $e_j = E_j^\tau(y)$ and $i_j = I_j^\tau(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. For $j,k=1,2, j\neq k$, solving (20) and (21) for the equilibrium yields the unique solution for τ-period utilization and investment,

$$E_j^\tau(y) = \frac{1 + \delta\overline{\eta}_j\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y \tag{22}$$

$$I_j^\tau(y) = \frac{\delta\overline{\eta}_j\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y. \tag{23}$$

Plugging (22) and (23) for the two sectors into the objective function in (19) yields

$$V_j^\tau(y) = (1 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}))\ln y + \Delta_\tau, \tag{24}$$

$$\equiv \kappa_\tau \ln y + \Psi_\tau, \tag{25}$$

where Δ_τ and Ψ_τ are constants that we ignore since they have no effect on the solution. Hence,

$$\kappa_\tau = 1 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \tag{26}$$

with, from (18), initial condition

$$\kappa_1 = 1 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}). \tag{27}$$

From (26) and (27), it follows that

$$\kappa_\tau = \sum_{t=0}^{\tau} \delta^t(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t. \tag{28}$$

Plugging (28) into (22) and (23) yields (10) and (12), respectively. Plugging (10) and (12) into $C_j^\tau(y) = E_j^\tau(y) - I_j^\tau(y)$ yields (11).

□

We now show that there is no disparity between the finite and infinite horizons. Specifically, using Proposition 3.1, we show that the limits of the equilibrium outcomes exist. In other words, there exists a unique equilibrium for the infinite horizon that is consistent with the sequence of finite-horizon equilibrium. We then use these limiting outcomes to derive the unique stationary distribution for capital. Proposition 3.2 provides the limits of the equilibrium outcomes.

Proposition 3.2. *Suppose that the investment goods are complementary. If $\overline{\eta}_1 + \overline{\eta}_2 + \alpha \in (0, 1)$, then for $j, k = 1, 2, j \neq k$, $\lim_{T\to\infty} E_j^T(y) \equiv E_j^\infty(y)$, $\lim_{T\to\infty} C_j^T(y) \equiv C_j^\infty(y)$, and $\lim_{T\to\infty} I_j^T(y) \equiv I_j^\infty(y)$ exist such that*

$$E_j^\infty(y) = \frac{1 - \delta(\overline{\eta}_k + \overline{\alpha})}{2 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{29}$$

$$C_j^\infty(y) = \frac{1 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})}{2 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{30}$$

$$I_j^\infty(y) = \frac{\delta\overline{\eta}_j}{2 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y. \tag{31}$$

Proof. Given that $\overline{\eta}_1 + \overline{\eta}_2 + \alpha \in (0, 1)$, taking limits of (10), (11) and (12) yields (29), (30) and (31), respectively. □

Using the limiting outcomes, Proposition 3.3 provides the stationary distribution of capital. Due to the fact that the equilibrium displays certainty equivalence, the stationary distribution depends directly on the means of the shocks. However, through (9), the stationary distribution of capital depends on the distribution of the shocks, i.e., first and higher moments.

Proposition 3.3. *Suppose that the investment goods are complementary. Then, the stationary distribution of capital is defined by*

$$\tilde{Y} = \left(\frac{\overline{\eta}_1^{\tilde{\eta}_1} \overline{\eta}_2^{\tilde{\eta}_2} \overline{\alpha}^{\tilde{\alpha}} \delta^{\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\alpha}}}{\left(2 - \delta\left(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}\right)\right)^{\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\alpha}}} \right)^{\frac{1}{1 - (\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\alpha})}}. \tag{32}$$

Proof. Plugging (29) and (31) into (9) and solving for $\tilde{Y} = \hat{y} = y$ yields (32). □

3.2 Substitutionary Investment Goods

When investment goods are substitutionary, the investment function is specified as

$$\phi(i_1, i_2, \boldsymbol{\eta}) = (i_1 + i_2)^{\eta}, \tag{33}$$

$\boldsymbol{\eta} \equiv \eta$. Using (33), (3) is rewritten as

$$\hat{y} = (i_1 + i_2)^{\eta} (y - e_1 - e_2)^{\alpha}, \tag{34}$$

The investment term $(i_1 + i_2)^{\eta}$ reflects the perfect substitutability of the sectors' investments.

Unlike the case of complementary investment goods, the equilibrium is not unique. In fact, when investment goods are substitutionary, there is a continuum of equilibrium that admits any allocation of the total investment between the two sectors but leaves total investment unchanged. The multiplicity of the equilibrium has no bearing on the dynamics of the capital and thus on sectors' future payoffs since, from (34), only total investment matters.

Proposition 3.4 states the properties of the equilibrium. The multiplicity of the equilibrium is reflected by the allocation of the investment goods between sectors 1 and 2. That is, for $j = 1, 2$, $\gamma_{1,\tau} \in [0, 1]$ is the fraction of total investment undertaken by sector j when the horizon is τ periods. Hence, $\gamma_{1,\tau} + \gamma_{2,\tau} = 1$.

Proposition 3.4. *Suppose that the investment goods are substitutionary. Then, there exists a continuum of recursive Cournot-Nash equilibrium for a T-period game, $T = 1, 2, \ldots$ For any equilibrium, for $\tau = 1, ..., T$,*

1. $C_1^\tau(y) = C_2^\tau(y)$.
2. *For $j = 1, 2$ and for any allocation $\{\gamma_{1,\tau}, \gamma_{2,\tau}\}$ such that $\gamma_{1,\tau}, \gamma_{2,\tau} \in [0, 1], \gamma_{1,\tau} + \gamma_{2,\tau} = 1$, $I_j^\tau(y) = \gamma_{j,\tau} \cdot (I_1^\tau(y) + I_2^\tau(y))$.*

Proof. See the proof of Proposition 3.5. □

Proposition 3.5 provides the utilization level as well as the production levels for consumption and investment goods corresponding to the equilibrium for any finite horizon. As in the case of complementary investment goods, the equilibrium displays certainty equivalence.

Proposition 3.5. *Suppose that the investment goods are substitutionary. Then, in equilibrium, for $\tau = 0, 1, \ldots, T$, for $j = 1, 2$, given an allocation $\{\gamma_{1,\tau}, \gamma_{2,\tau}\}$, sector j utilizes*

$$E_j^\tau(y) = \frac{1 + \gamma_{j,\tau}\delta\overline{\eta}\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)}{2 + \delta(\overline{\eta} + \overline{\alpha})\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)} y \tag{35}$$

units of capital for the production of

$$C_j^\tau(y) = \frac{1}{2 + \delta(\overline{\eta} + \overline{\alpha})\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)} y \tag{36}$$

units of consumption goods and

$$I_j^{\tau}(y) = \frac{\gamma_{j,\tau}\delta\overline{\eta}\left(\sum_{t=0}^{\tau-1}\delta^t\left(\overline{\eta}+\overline{\alpha}\right)^t\right)}{2+\delta(\overline{\eta}+\overline{\alpha})\left(\sum_{t=0}^{\tau-1}\delta^t(\overline{\eta}+\overline{\alpha})^t\right)}y \tag{37}$$

units of investment goods.

Proof. We first derive utilization, investment, and value functions in the one-period horizon. We then consider a τ-period horizon and solve for utilization, investment and value functions recursively. We finally impose the initial condition given by the one-period-horizon solution.

1. Consider first the one-period horizon. Using (5), (6) and (34), for $j, k = 1, 2, j \neq k$, given $\{E_k^1(y), I_k^1(y)\}$, sector j's one-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^1(y), I_j^1(y)\} = &\arg\max_{e_j, i_j}\left\{\ln(e_j - i_j) + \delta\overline{\eta}\ln(i_j + I_k^1(y))\right.\\ &\left.+\delta\overline{\alpha}\ln(y - e_j - E_k^1(y)) - \delta\ln 2\right\}.\end{aligned} \tag{38}$$

The first-order conditions corresponding to (38) are

$$e_j: \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}}{y - e_j - E_k^1(y)}, \tag{39}$$

$$i_j: \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}}{i_j + I_k^1(y)}, \tag{40}$$

evaluated at $e_j = E_j^1(y)$ and $i_j = I_j^1(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. However, individual investment cannot be determined because $I_j^1(y)$ and $I_k^1(y)$ have an effect on equilibrium condition only through their sum. To see this, for $j = 1, 2$, plugging $C_j^1(y) = E_j^1(y) - I_j^1(y)$ into (39) and (40) and rearranging yields the system

$$\frac{1}{C_1^1(y)} = \frac{\delta\overline{\alpha}}{y - C_1^1(y) - I_1^1(y) - C_2^1(y) - I_2^1(y)}, \tag{41}$$

$$\frac{\delta\overline{\eta}}{I_1^1(y) + I_2^1(y)} = \frac{\delta\overline{\alpha}}{y - C_1^1(y) - I_1^1(y) - C_2^1(y) - I_2^1(y)}, \tag{42}$$

$$\frac{1}{C_2^1(y)} = \frac{\delta\overline{\eta}}{y - C_2^1(y) - I_2^1(y) - C_1^1(y) - I_1^1(y)}, \tag{43}$$

$$\frac{\delta\overline{\eta}}{I_1^1(y) + I_2^1(y)} = \frac{\delta\overline{\alpha}}{y - C_2^1(y) - I_2^1(y) - C_1^1(y) - I_1^1(y)}, \tag{44}$$

which defines the one-period-horizon solution for the equilibrium, i.e., $\{C_j^1(y), I_j^1(y)\}_{j=1}^2$. From (42) and (44), one equation is redundant, which implies that there are three equations for four unknowns. In fact, $C_j^1(y)$, $C_k^1(y)$ and $I_k^1(y) + I_j^1(y)$ have unique solutions, but $I_j^1(y)$ and $I_k^1(y)$ cannot be determined separately.

Letting $\gamma_{j,1} \in (0,1)$ be the fraction of total investment goods produced by sector j in the one-period horizon, solving (39) and (40) for the equilibrium yields the solution for one-period-horizon utilization and investment:

$$E_j^1(y) = \frac{1 + \gamma_{j,1}\delta\overline{\eta}}{2 + \delta\left(\overline{\eta} + \overline{\alpha}\right)} y \tag{45}$$

$$I_j^1(y) = \gamma_{j,1} \frac{\delta\overline{\eta}}{2 + \delta\left(\overline{\eta} + \overline{\alpha}\right)} y. \tag{46}$$

Plugging (45) and (46) for the two sectors into the objective function in (38) yields

$$V_j^1(y) = \left(1 + \delta\left(\overline{\eta} + \overline{\alpha}\right)\right) \ln y + \Psi_1 \tag{47}$$

where Ψ_1 is a constant for the one-period horizon that has no effect on the solution.

2. Having solved for the one-period-horizon, we consider next a τ-period-horizon for which the continuation value function is of the form $V^{\tau-1}(y) = \kappa_{\tau-1} \ln y + \Psi_{\tau-1}$ where $\kappa_{\tau-1}$ and $\Psi_{\tau-1}$ are unknown constants. For $j, k = 1, 2, j \neq k$, given $V^{\tau-1}(y) = \kappa_{\tau-1} \ln y + \Psi_{\tau-1}$ and $\{E_k^\tau(y), I_k^\tau(y)\}$,

sector j's τ-period-horizon optimal policies satisfy

$$\{E_j^\tau(y), I_j^\tau(y)\} = \arg\max_{e_j, i_j} \{\ln(e_j - i_j) + \delta\overline{\eta}\kappa_{\tau-1}\ln(i_j + I_k^\tau(y)) \\ +\delta\overline{\alpha}\kappa_{\tau-1}\ln(y - e_j - E_k^\tau(y)) + \delta\Psi_{\tau-1}\}. \quad (48)$$

The first-order conditions corresponding to (48) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}\kappa_{\tau-1}}{y - e_j - E_k^\tau(y)}, \quad (49)$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}\kappa_{\tau-1}}{i_j + I_k^\tau(y)} \quad (50)$$

evaluated at $e_j = E_j^\tau(y)$ and $i_j = I_j^\tau(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. However, as noted in the one-period-horizon, individual investment cannot be determined because $I_j^1(y)$ and $I_k^1(y)$ have an effect on equilibrium condition only through their sum. Letting $\gamma_{j,\tau} \in (0,1)$ be the fraction of total investment goods produced by sector j in the τ-period horizon, Solving (49) and (50) for the equilibrium yields the solution for utilization and investment,

$$E_j^\tau(y) = \frac{1 + \gamma_{\tau,j}\delta\overline{\eta}\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta} + \overline{\alpha})} y \quad (51)$$

$$I_j^\tau(y) = \frac{\gamma_{\tau,j}\delta\overline{\eta}\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta} + \overline{\alpha})} y. \quad (52)$$

Plugging (51) and (52) for the two sectors into the objective function in (48) yields

$$V_j^\tau(y) = (1 + \delta\kappa_{\tau-1}(\overline{\eta} + \overline{\alpha}))\ln y + \Delta_\tau \quad (53)$$

$$\equiv \kappa_\tau \ln y + \Psi_\tau, \quad (54)$$

where Δ_τ and Ψ_τ are constants that we ignore since they have no effect on the solution. Hence,

$$\kappa_\tau = 1 + \delta\kappa_{\tau-1}\left(\overline{\eta} + \overline{\alpha}\right) \tag{55}$$

with, from (47), initial condition

$$\kappa_1 = 1 + \delta\left(\overline{\eta} + \overline{\alpha}\right). \tag{56}$$

From (55) and (56), it follows that

$$\kappa_\tau = \sum_{t=0}^{\tau} \delta^t (\overline{\eta} + \overline{\alpha})^t. \tag{57}$$

Plugging (57) into (51) and (52) yields (35) and (37). Plugging (35) and (37) into $C_j^\tau(y) = E_j^\tau(y) - I_j^\tau(y)$ yields (36).

□

For each point in the continuum of finite-horizon equilibrium, the limits to the finite-horizon equilibrium outcomes exist. As in the case of complementary investment goods, the case of substitutionary investment goods yields no disparity between the finite and infinite horizons. Proposition 3.6 provides the equilibrium for an infinite horizon, i.e., the limits of the equilibrium outcomes in Proposition 3.5.

Proposition 3.6. *Suppose that the investment goods are substitutionary. If* $\eta + \alpha \in (0, 1)$*, then for* $j = 1, 2$*,* $\lim_{T\to\infty} E_j^T(y) \equiv E_j^\infty(y)$*,* $\lim_{T\to\infty} C_j^T(y) \equiv C_j^\infty(y)$*, and* $\lim_{T\to\infty} I_j^T(y) \equiv I_j^\infty(y)$ *exist such that, given an allocation* $\{\gamma_{1,\infty}, \gamma_{2,\infty}\}$*,*

$$E_j^\infty(y) = \frac{1 - \delta\left((1-\gamma_{j,\infty})\overline{\eta} + \overline{\alpha}\right)}{2 - \delta\left(\overline{\eta} + \overline{\alpha}\right)} y \tag{58}$$

$$C_j^\infty(y) = \frac{1 - \delta\left(\overline{\eta} + \overline{\alpha}\right)}{2 - \delta\left(\overline{\eta} + \overline{\alpha}\right)} y \tag{59}$$

$$I_j^\infty(y) = \frac{\gamma_{j,\infty}\delta\overline{\eta}}{2 - \delta\left(\overline{\eta} + \overline{\alpha}\right)} y. \tag{60}$$

Proof. Given that $\eta + \alpha \in (0, 1)$, taking limits of (35), (36), and (37) yields (58), (59) and (60). □

Although the equilibrium is a continuum, the perfect substitutability of the investment goods implies a unique stationary distribution of capital. Hence,

Proposition 3.7. *Suppose that the investment goods are substitutionary. Then, the stationary distribution of capital is defined by*

$$\tilde{Y} = \left(\frac{\delta^{\tilde{\eta}+\tilde{\alpha}} \overline{\eta}^{\tilde{\eta}} \overline{\alpha}^{\tilde{\alpha}}}{\left(2 - \delta \left(\overline{\eta} + \overline{\alpha}\right)\right)^{\tilde{\eta}+\tilde{\alpha}}} \right)^{\frac{1}{1-(\tilde{\eta}+\tilde{\alpha})}}. \tag{61}$$

Proof. Plugging (58) and (60) into (34) and solving for $\tilde{Y} = \hat{y} = y$ yields (61). □

Before proceeding with the comparison between the Cournot-Nash equilibrium and the solution of the social planner, we compare differences between complements and substitutes. Apart from the uniqueness property, by comparing Propositions 3.1 and 3.2 with Propositions 3.5 and 3.6, the policy functions for the sectors' behavior are of similar form. Similarly, by comparing Propositions 3.3 and 3.7, the functional form of the stationary distribution is robust to the type of investment goods. Regardless of the belief one can have about the type of investment goods that affect non-excludable capital, the equilibrium outcomes and the stationary distribution for capital is robust (in terms of functional forms) to different types of investment goods.

4 Investment Externality

Having characterized the recursive Cournot-Nash equilibrium, we study the effect of the investment externality (combined with the dynamic externality) on behavior and the stationary distribution of capital. To that end, we first provide the optimal solution of a social planner in the infinite horizon case.[2] We then characterize the tragedy of the commons when the utilization of

[2]To simplify the discussion, we omit the finite-horizon case. Our results on the tragedy of the commons hold for any finite horizon.

capital produces both consumption and investment goods. We finally derive the stationary distributions of capital corresponding to the social planner's optimal solution and compare them with the stationary distributions corresponding to the recursive Cournot-Nash equilibrium.

Under social planning, the infinite-horizon value function of the social planner satisfies

$$W^{\infty}(y) = \max_{\{e_j, i_j\}_{j=1}^2} \left\{ \ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta \mathbb{E} W^{\infty}(g(i_1, i_2, \tilde{\eta}) \cdot (y - e_1 - e_2)^{\tilde{\alpha}}) \right\}, \tag{62}$$

where $g(i_1, i_2, \tilde{\eta}) = i_1^{\tilde{\eta}_1} i_2^{\tilde{\eta}_2}$ if investment goods are complements and $g(i_1, i_2, \tilde{\eta}) = (i_1 + i_2)^{\tilde{\eta}}$ if substitutes. For $j = 1, 2$, let $E_j^{*\infty}(y)$, $C_j^{*\infty}(y)$, and $I_j^{*\infty}(y)$ be the optimal solutions for utilization, consumption and investment where the symbol $*$ distinguishes optimal behavior from behavior in the Cournot-Nash equilibrium.

Proposition 4.1 provides the social planner's optimal solution for total utilization, and total production of consumption and investment goods under complementary and substitutionary investment goods. The introduction of the game has an effect on the comparative analysis. Under social planning with either complementary or substitutionary investment goods, there is *separation* in the sense that the investment shock $\boldsymbol{\eta}$ has no effect on total utilization whereas the shock α has no effect on total investment. However, with a game, from Propositions 3.2 and 3.6, an increase in any of the means of the investment shocks decreases total utilization and an increase in $\overline{\alpha}$ causes total investment to increase.

Proposition 4.1. *There exists a unique optimal solution to* (62)*.*

1. *Suppose that investment goods are complementary. If* $\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha} \in (0, 1)$*, then*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) = (1 - \delta\overline{\alpha})y, \tag{63}$$

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) = (1 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}))y, \tag{64}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \delta(\overline{\eta}_1 + \overline{\eta}_2)y. \tag{65}$$

2. *Suppose that investment goods are substitutionary. If* $\overline{\eta} + \overline{\alpha} \in (0, 1)$*, then*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) = (1 - \delta\overline{\alpha})y, \tag{66}$$

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) = (1 - \delta(\overline{\eta} + \overline{\alpha}))y, \tag{67}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \delta\overline{\eta}y. \tag{68}$$

Proof. See Appendix A. □

Whether the investment goods are complementary or substitutionary, the investment externality yields a tragedy in the commons in the following sense. Under a game, total utilization increases. Moreover, the production of consumption goods increases at the expense of investment goods.

Proposition 4.2. *Suppose that investment goods are either complementary or substitutionary. Then,*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) < E_1^{\infty}(y) + E_2^{\infty}(y), \tag{69}$$

and

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) < C_1^{\infty}(y) + C_2^{\infty}(y), \tag{70}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) > I_1^{\infty}(y) + I_2^{\infty}(y). \tag{71}$$

Proof. Comparing Propositions 3.2, 3.6, and 4.1 yields inequalities (69), (70), and (71). □

The investment externality has an effect on the stationary distribution of capital as well. Proposition 4.3 provides the stationary distribution of capital under social planning.

Proposition 4.3. *Under social planning, the stationary distribution of capital is unique.*

1. *If investment goods are complementary, then*

$$Y^* = \left(\overline{\eta}_1^{\tilde{\eta}_1}\overline{\eta}_2^{\tilde{\eta}_2}\overline{\alpha}^{\tilde{\alpha}}\delta^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha}}\right)^{\frac{1}{1-(\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha})}}. \tag{72}$$

2. *If investment goods are substitutionary, then*

$$Y^* = \left(\overline{\eta}^{\tilde{\eta}}\overline{\alpha}^{\tilde{\alpha}}\delta^{\tilde{\eta}+\tilde{\alpha}}\right)^{\frac{1}{1-(\tilde{\eta}+\tilde{\alpha})}}. \tag{73}$$

Proof. If investment goods are complementary, then plugging (63), (81), and (82) into (9) and solving for $\tilde{Y}^* = \hat{y} = y$ yields (72). Next, if investment goods are substitutionary, then plugging (66) and (68) into (34) and solving for $\tilde{Y}^* = \hat{y} = y$ yields (73). □

Regardless of the type of investment goods, the effect of the investment externality on the stationary distribution is illustrated in Figure 1.[3] The two solid concave lines depict expression (2) evaluated at the highest and lowest value of the realizations of the random shocks under social planning, i.e.,

$$y_{t+1} = g(I_1^{*\infty}(y_t), I_2^{*\infty}(y_t), \boldsymbol{\eta}) \cdot f(y_t - E_1^{*\infty}(y_t) - E_2^{*\infty}(y_t), \alpha). \tag{74}$$

The two dotted concave lines also depict expression (2) evaluated at the highest and lowest value of the realizations of the random shocks but under a game, i.e.,

$$y_{t+1} = g(I_1^{\infty}(y_t), I_2^{\infty}(y_t), \boldsymbol{\eta}) \cdot f(y_t - E_1^{\infty}(y_t) - E_2^{\infty}(y_t), \alpha). \tag{75}$$

The intersection of these lines with the 45 degree line defines the end-points of the stationary distributions under social planning and under a game.[4] Specifically, the stationary distribution under social planning has support $[Y^*_{\min}, Y^*_{\max}]$ whereas the stationary distribution under a game has support $[Y_{\min}, Y_{\max}]$. Since $Y_{\min} < Y^*_{\min}$ and $Y_{\max} < Y^*_{\max}$, the effect of the game with

[3] When investment goods are complementary, compare (32) and (72). With substitutes, compare (61) and (73).

[4] Recall that in our model investment is required to maintain the capital. Without investment, the stationary distribution is degenerate at zero.

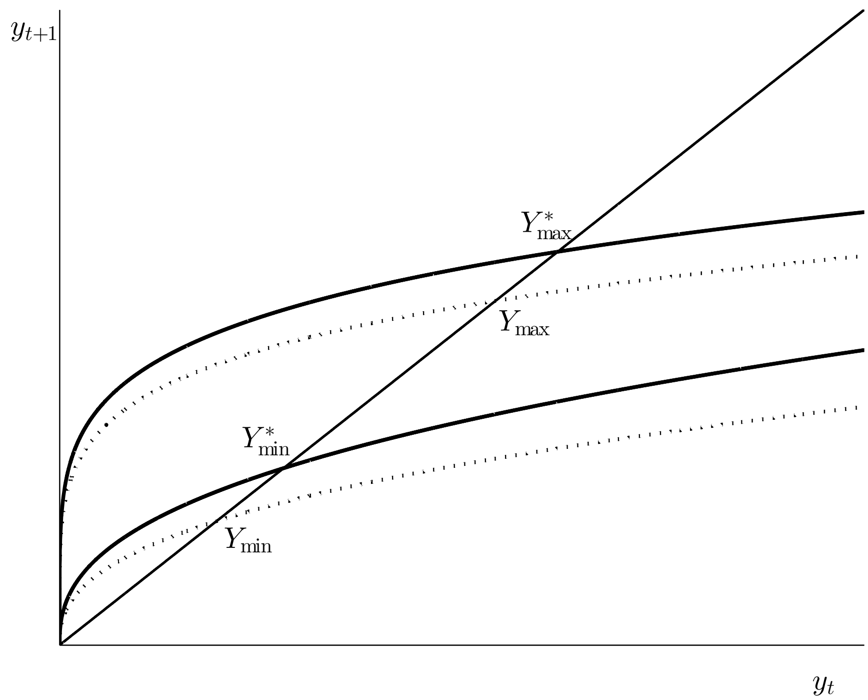

Figure 1: The Effect of the Investment Externality on the Stationary Distribution

an investment externality is to reduce the effectiveness of the stock of non-excludable capital. However, it is ambiguous whether the negative effect is strongest with complements or substitutes, i.e., it depends on the values of the parameters.

5 Final Remarks

In order to study the effect of the investment externality on utilization, production and the dynamic path of non-excludable capital, we have considered a stochastic environment in which agents know the true distribution of the random shocks. However, agents generally face more than just uncertainty in outcomes since the true distributions of these shocks are never known exactly.

In other words, agents generally face *structural uncertainty* because they do not know the structure of the economy. The issue of structural uncertainty in a dynamic game with an investment externality is studied in a companion paper (Mirman and Santugini, 2013). Unlike uncertainty in outcomes, structural uncertainty evolves through learning. In that case, agents make utilization and production decisions as well as learn simultaneously about the stochastic process. Although the characterization of a dynamic game with Bayesian dynamics (and without the assumption of adaptive learning) is generally intractable, we characterize the symmetric Bayesian-learning recursive Cournot-Nash equilibrium. The addition of learning to a stochastic environment is shown to have a profound effect on the equilibrium since decision-making and learning are nonseparable and influence each other.

A Optimal Solution of the Social Planner

In this appendix, we derive the social planner's optimal solution in the case of complementary and substitutionary investment goods. We consider the infinite horizon by conjecturing that the value function is of the form $W^{\infty}(y) = \kappa_{\infty} \ln y + \Psi_{\infty}$. As noted, the linear conjecture can be inferred by solving the problem recursively.

Given $W^{\infty}(y) = \kappa_{\infty} \ln y + \Psi_{\infty}$, (62) is rewritten as

$$\begin{aligned} W^{\infty}(y) = & \max_{\{e_j, i_j\}_{j=1}^{2}} \{\ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta\kappa_{\infty}\overline{\eta}_1 \ln i_1 + \delta\kappa_{\infty}\overline{\eta}_2 \ln i_2 \\ & + \delta\kappa_{\infty}\overline{\alpha} \ln(y - e_1 - e_2) + \delta\Psi_{\infty}\} \end{aligned} \tag{76}$$

if the investment goods are complementary and

$$\begin{aligned} W^{\infty}(y) = & \max_{\{e_j, i_j\}_{j=1}^{2}} \{\ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta\kappa_{\infty}\overline{\eta} \ln(i_1 + i_2) \\ & + \delta\kappa_{\infty}\overline{\alpha} \ln(y - e_1 - e_2) + \delta\Psi_{\infty}\} \end{aligned} \tag{77}$$

if the investment goods are substitutionary.

For complements, for $j = 1, 2$, the first-order conditions corresponding to (76) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_{\infty}\overline{\alpha}}{y - e_1 - e_2}, \tag{78}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_{\infty}\overline{\eta}_j}{i_j}, \tag{79}$$

which yields

$$E_j^{*\infty}(y) = \frac{1 + \delta\kappa_{\infty}\overline{\eta}_j}{2 + \delta\kappa_{\infty}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{80}$$

$$I_j^{*\infty}(y) = \frac{\delta\kappa_{\infty}\overline{\eta}_j}{2 + \delta\kappa_{\infty}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y. \tag{81}$$

Plugging (80) and (81) back into (76) implies that

$$\kappa_\infty = \frac{2}{1-\delta(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})}. \tag{82}$$

Plugging (82) into (80) and (81) and summing over j yields (63) and (65). Plugging (63) and (65) into $\sum_{j=1}^2 C_j^{*\infty}(y) = \sum_{j=1}^2 (E_j^{*\infty}(y) - I_j^{*\infty}(y))$ yields (64).

For substitutes, for $j = 1, 2$, the first-order conditions corresponding to (77) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_\infty\overline{\alpha}}{y - e_1 - e_2} \tag{83}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_\infty\overline{\eta}}{i_1 + i_2}, \tag{84}$$

which yields

$$E_j^{*\infty}(y) = \frac{2+\delta\kappa_\infty\overline{\eta}}{4+2\delta\kappa_\infty(\overline{\eta}+\overline{\alpha})}y \tag{85}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \frac{\delta\kappa_\infty\overline{\eta}}{2+\delta\kappa_\infty(\overline{\eta}+\overline{\alpha})}y \tag{86}$$

since the social planner only needs to solve for total investment. Plugging (85) and (86) back into (77) yields

$$\kappa_\infty = \frac{2}{1-\delta(\overline{\eta}+\overline{\alpha})}. \tag{87}$$

Plugging (87) into (85) and summing over j yields (66). Plugging (87) into (86) yields (68). Plugging (66) and (68) into $\sum_{j=1}^2 C_j^{*\infty}(y) = \sum_{j=1}^2 (E_j^{*\infty}(y) - I_j^{*\infty}(y))$ yields (67).

References

W.A. Brock and L.J. Mirman. Optimal Economic Growth and Uncertainty: The Discounted Case. *J. Econ. Theory*, 4(3):479–513, 1972.

D. Cass. Optimum Growth in an Aggregative Model of Capital Accumulation. *Rev. Econ. Stud.*, 32(3):233–240, 1965.

T.C. Koopmans. On the Concept of Economic Growth. In *Semaine d'Étude sur le Rôle de l'Analyse Économétrique dans la Formulation de Plans de Développement*, volume 28, pages 225–300. Pontificau Academiae Scientiarum Scripta Varia, Vatican, 1965.

C. Koulovatianos and L.J. Mirman. The Effects of Market Structure on Industry Growth: Rivalrous Non-Excludable Capital. *J. Econ. Theory*, 133(1):199–218, 2007.

D. Levhari and L.J. Mirman. The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution. *Bell J. Econ.*, 11(1):322–334, 1980.

L.J. Mirman. On the Existence of Steady State Measures for One Sector Growth Models with Uncertain Technology. *Int. Econ. Rev.*, 13(2):271–286, 1972.

L.J. Mirman. The Steady State Behavior of a Class of One Sector Growth Models with Uncertain Technology. *J. Econ. Theory*, 6(3):219–242, 1973.

L.J. Mirman. Dynamic Models of Fishing: A Heuristic Approach. In P.-T. Liu and J.G. Sutinen, editors, *Control Theory in Mathematical Economics*, pages 39–73. Marcel Dekker, 1979.

L.J. Mirman and M. Santugini. Learning and Technological Progress in Dynamic Games. *Dynam. Games Appl.*, 2013. Forthcoming.

L.J. Mirman and I. Zilcha. On Optimal Growth under Uncertainty. *J. Econ. Theory*, 11(3):329–339, 1975.

R.M. Solow. A Contribution to the Theory of Economic Growth. *Quart. J. Econ.*, 70(1):65–94, 1956.