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Changing Income in an Uncertain Environment

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Abstract:
We study the effect of changing income on optimal decisions in the multidimensional expected utility framework. Using the KM utility representation, we show that the comparative analysis under uncertainty is founded on classical demand theory under certainty and is linked to the effect of changing risk aversion, which also depends on classical demand theory.

Keywords: Classical Demand Theory, Consumption-Saving Problem, Income, Risk Aversion, Uncertainty

JEL Classification: D01, D81, D91

1 Introduction

The use of comparative statics is at the foundation of the study of behavior in economics. Under certainty, the income and substitution effects are building blocks for the understanding of the effects of parameter changes on optimal behavior. However, the notions of income and substitution effects have not had the same impact on the comparative statics for optimal behavior under uncertainty. The reason is that comparative statics under uncertainty began with Arrow (1965) and Pratt (1964) in the portfolio problem with a one-dimensional utility function.[1] In their papers (as well as all applications with a one-dimensional utility function in the literature), there is no income effect. Hence, the comparative analysis in the portfolio problem depends only on the substitution effect. This is not the case with a multidimensional utility function (i.e., several goods) as shown by Kihlstrom and Mirman (1974) (KM). It is in KM that the notions of income and substitution effects are first used to study the effect of risk aversion in the multidimensional case. Recently, Mirman and Santugini (2013) (MS) used the income and substitution effects to study the behavior of optimal decisions due to changes in risk aversion in a general multidimensional setting under uncertainty.

Since virtually all aspects of behavior have to deal with uncertainty (e.g., consumers face uncertainty about prices and income.), it is important to understand the comparative analysis under uncertainty. Moreover, although the comparative analysis in the portfolio problem under uncertainty in the one-dimensional case is a natural first step, the comparative statics properties in the general multidimensional expected utility framework must be studied. In the multidimensional case, the effect of uncertainty on optimal behavior combines both tastes (i.e., ordinal preferences) and attitudes toward risk (i.e., risk aversion). In order to study the effect of risk aversion on optimal behavior, the effects of tastes and risk aversion must be identified. This is done by KM which generalizes the notion of risk aversion to the multidimensional case by introducing utility representations that are concave

[1] Specifically, Arrow (1965) considers the effect of changing income in the portfolio problem. Pratt (1964), on the other hand, deals with changes in risk aversion on optimal decisions.

transformations of each other. It is precisely this definition of risk aversion that MS used to characterize the effect of risk aversion on optimal behavior.

It is the purpose of this paper to study the effect of changing income on optimal decisions in the multidimensional expected utility framework. To that end, we use the KM utility representation to highlight the role of risk aversion on the comparative analysis. Indeed, the introduction of uncertainty is always implicitly accompanied by assumptions regarding attitudes toward risk. In the general utility representation, it is not very difficult to do comparative statics and obtain income and substitution expressions that are expectations of the classical income and substitution effects. However, these expectations do not reveal the role played by risk aversion in determining optimal behavior. In other words, with the general utility representation, risk aversion is implicit.

The general utility representation is thus inappropriate to present comparative statics results under uncertainty since it hides the role of risk aversion. In order to obtain the proper effect of changing income, we use the KM utility representation to make the role played by risk aversion explicit. Using the KM utility representation, we show that the comparative analysis under uncertainty is founded on classical demand theory under certainty and is linked to the effect of changing risk aversion, which also depends on classical demand theory.

More specifically, we study the effect of changing income in the consumption-saving problem when the rate of return is random. We first decompose the effect of a change in income on both the sure good and the risky good. We show that the effect of changing income depends on the now explicit change in risk aversion. Specifically, the effect of changing income can be decomposed into a modified income effect and a hybrid effect. The modified income effect captures the effect of changing income through uncertainty and the hybrid effect captures the effect of changing risk aversion due to changes in income. The hybrid effects are related to the pure risk aversion effect contained in MS.

Using the decomposition, we then study the direction of the effect of a change in income. In general, the sign of the modified income effect depends on the normality of the goods whereas the sign of the hybrid effect depends on both tastes and attitudes toward risk. We consider both constant and decreasing absolute risk aversion. Whether risk preferences exhibit constant or decreasing absolute risk aversion, it is shown that an increase in income always increases the amount of the sure good. However, the effect of changing income has an ambiguous effect on the risky good. In particular, suppose that both goods are normal and risk preferences exhibit decreasing absolute risk aversion. On the one hand, the normality of the risky good induces more consumption of the risky good when income increases. On the other hand, as income increases, the individual becomes less risk averse, which, under certain conditions regarding the income and substitution effects, induces less consumption of the risky good. Hence, the overall effect of increasing income is ambiguous. One implication of our results is that there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965) in the multidimensional case.

Finally, we discuss the appropriateness of an alternative approach to study risk aversion suggested by Selden (1978), which has been widely popularized through the parametric model of Epstein and Zin (1989) (henceforth, the Selden-EZ approach). We show that the Selden-EZ approach cannot be used to isolate the effect of risk aversion. In particular, the comparative analysis under the Selden-EZ approach yields outcomes that are inconsistent with ordinal preferences.

The paper is organized as follows. In Section 2, we introduce the model and discuss the KM approach. Section 3 presents the comparative analysis. Section 4 concludes the paper by discussing the Selden-EZ approach.

2 KM Framework

The effect of uncertainty on optimal behavior in the multidimensional case combines both tastes (i.e., ordinal preferences) and attitudes toward risk (i.e., risk aversion). In order to study the effect of risk aversion on optimal behavior, the effects of tastes and risk aversion must be identified. This issue does not arise for the class of one-dimensional strictly increasing utility functions since tastes are represented by the natural ordering on the real line, i.e., $x_A > x_B$ means that $x_A \succ x_B$. However, the relationship between the utility representation, uncertainty, risk aversion, and tastes is much more delicate in the multidimensional case since there is no natural order. In other words, different utility functions incorporate different tastes as well as different attitudes toward risk so that the link between risk aversion and risk averse behavior cannot be clearly identified.[2]

In this section, we use the approach established by KM for the study of the effect of risk aversion on optimal behavior under uncertainty in the multidimensional case. When going from certainty to uncertainty, risk aversion is always implicit. Hence, uncertainty and risk aversion are naturally entangled with each other. In other words, the effect of uncertainty on behavior cannot be studied without taking account of the role played by risk aversion.

To show the intricacies of optimal behavior under uncertainty, it is useful to begin with a general utility representation. In that case, risk aversion, implicit in optimal behavior, cannot in general be recognized. To remedy that problem, we consider utility functions that are concave transformations of each other as studied in KM. Studying optimal behavior under uncertainty using the KM framework makes the role of risk aversion explicit. Moreover, this representation clarifies the relationship between uncertainty and risk aversion as well as their distinct roles on optimal behavior. We apply the KM framework to the consumption-saving problem under uncertainty and derive optimal behavior. In the next section, we highlight the role of risk aversion for the effect of changing income on optimal behavior in the consumption-

[2]For instance, KM provides an example in which the preference between a sure outcome and a gamble depends solely on tastes and not on risk aversion. See Appendix A.

saving problem under uncertainty.

General Utility Representation. Consider an individual making decisions under uncertainty. Let the consumption profile $(x, \tilde{y}) \in \mathbb{R}_+^2$ have utility representation $V(x, \tilde{y})$. In the stochastic environment, x is the *sure* good and $\tilde{y}$ is the *risky* good due to the presence of randomness in the budget constraint. Specifically, the maximization problem under uncertainty is

$$\max_x \mathbb{E}_{\tilde{\varepsilon}} V(x, Z(x, \tilde{\varepsilon}, I)), \tag{1}$$

where $\mathbb{E}_{\tilde{\varepsilon}}$ is the expectation operator over a random shock $\tilde{\varepsilon}$. The risky good depends on the sure good x, the random shock $\tilde{\varepsilon}$ and the income I through a budget constraint, i.e., $\tilde{y} = Z(x, \tilde{\varepsilon}, I)$. Assuming that the second-order condition is satisfied, optimal consumption is defined by the first-order condition corresponding to (1), i.e.,

$$\mathbb{E}_{\tilde{\varepsilon}} \left[\frac{\partial V(x, Z(x, \tilde{\varepsilon}, I))}{\partial x} + \frac{\partial V(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial x} \right] = 0 \tag{2}$$

evaluated at $x = x^*$.

Although implicit, Expression (2) is uninformative regarding the effect of risk aversion on optimal behavior under uncertainty. In particular, the general utility representation confounds risk aversion and tastes. Indeed, consider the effect of changing income on the sure good. That is, using (2),

$$\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathbb{E}_{\tilde{\varepsilon}} \left[\frac{\partial^2 U(x, Z(x, \tilde{\varepsilon}, I))}{\partial x \partial \tilde{y}} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial I} + \frac{\partial U(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \frac{\partial^2 Z(x, \tilde{\varepsilon}, I)}{\partial x \partial I} \right] \tag{3}$$

where $\stackrel{S}{=}$ means *of the same sign as*. Expression (3) does not make risk aversion explicit and provides no information on how risk aversion influences the comparative analysis. In fact, from expression (3), it appears that uncertainty has minimal effect on optimal behavior, i.e., removing the expectation operator in (3) yields the effect of changing income on the sure good in a deterministic environment. It looks as if there is no risk aversion effect although the introduction of uncertainty cannot occur without regard to risk aversion. Hence, the general utility representation hides the intricacies of optimal behavior under uncertainty.

KM utility representation. In order to clarify the relationship between uncertainty and risk aversion as well as their distinct roles on optimal behavior, the KM utility representation is adopted. Formally, let $V(x,\tilde{y}) \equiv \varphi(U(x,\tilde{y}))$ be the utility associated with the consumption profile $(x,y) \in \mathbb{R}^2_+$. Here, φ is a strictly increasing and concave function, $\varphi' > 0, \varphi'' \leq 0$ and $U(x,\tilde{y})$ is a quasiconcave function. Under the KM utility representation, tastes and attitudes toward risk are not confounded. Indeed, $U(x,\tilde{y})$ refers to tastes as well as attitudes toward risk whereas φ reflects *changes* in risk aversion. Specifically, a more concave φ (and, thus, a more concave V) means that the individual is more risk-averse. Concave transformations of the utility function alter the expected marginal rate of substitution in a way that is consistent with ordinal preferences, but do not alter the deterministic marginal rate of substitution. In particular, with the KM approach, attitudes towards risk (i.e., the concavity of φ) is independent of any gamble.

Using the KM utility representation, the maximization problem under uncertainty defined by (1) is rewritten as

$$\max_x \mathbb{E}_{\tilde{\varepsilon}}\varphi\left(U(x,Z(x,\tilde{\varepsilon},I))\right). \tag{4}$$

Using (4), optimal consumption is defined by the first-order condition the first-order condition

$$\mathbb{E}_{\tilde{\varepsilon}}\varphi'\left(U(x,Z(x,\tilde{\varepsilon},I))\right)\cdot MU(x,Z(x,\tilde{\varepsilon},I)) = 0 \tag{5}$$

evaluated at $x = x^*$. Here,

$$MU(x,Z(x,\tilde{\varepsilon},I)) \equiv \frac{\partial U(x,Z(x,\tilde{\varepsilon},I))}{\partial x} + \frac{\partial U(x,Z(x,\tilde{\varepsilon},I))}{\partial \tilde{y}}\frac{\partial Z(x,\tilde{\varepsilon},I)}{\partial x} \tag{6}$$

is the marginal utility of consumption for the sure good. Unlike (2), expression (5) makes risk aversion explicit through the term $\varphi'\left(U(x,Z(x,\tilde{\varepsilon},I))\right)$. It is also clear that risk aversion and tastes are entwined. In particular, changing income has an effect on both attitudes toward risk and the marginal utility of consumption. That is, for the sure good, using (5),

$$\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathbb{E}_{\tilde{\varepsilon}}\varphi''\left(U(x, Z(x, \tilde{\varepsilon}, I))\right) \cdot MU(x, Z(x, \tilde{\varepsilon}, I))^2 + \varphi'\left(U(x, Z(x, \tilde{\varepsilon}, I))\right) \cdot \frac{\partial MU(x, Z(x, \tilde{\varepsilon}, I))}{\partial I}. \tag{7}$$

Unlike expression (3), expression (7) highlights the importance of the role of risk aversion for the comparative analysis through the concavity of φ. However, while (7) is more informative than (3), it remains to analyze the influence of attitudes toward risk (e.g., constant or decreasing absolute risk aversion) and tastes (e.g., normal or income-neutral goods) on the comparative analysis. To that end, we turn to a classical application.

Application to Consumption-Saving Problem. We now apply the KM approach to the consumption-saving problem under uncertainty. We present the model and derive optimal behavior. In the next section, we perform a comparative analysis of the effect of changing income on optimal behavior.

Consider an individual making consumption and saving decisions under uncertainty. As noted, in the stochastic environment, x is the sure good and $\tilde{y}$ is the risky good due to the presence of randomness in the budget constraint. Specifically, the individual is endowed with income $I > 0$ from which $x \in (0, I)$ is consumed and the remaining $s \equiv I - x$ is saved in a risky asset with the random gross return $\tilde{R}$. Given the random budget constraint, it follows that $\tilde{y} = \tilde{R}(I - x)$.

To facilitate the discussion, we adopt a binary distribution for the rate of return of the risky asset and consider additive preferences.

Assumption 2.1. $\tilde{R} \sim (\pi \circ \overline{R}, (1-\pi) \circ \underline{R})$ *such that* $\pi \in (0, 1)$ *and* $0 < \underline{R} < \overline{R}$.

Assumption 2.2. $U(x, \tilde{y}) = u_1(x) + u_2(\tilde{y})$ *such that* $u_1', u_2' > 0, u_1'', u_2'' \leq 0$.

Additive preferences allows us to study several types of tastes. If $u_1'', u_2'' < 0$, then both goods are normal. If $u_1'' = 0, u_2'' < 0$ or $u_1'' < 0, u_2'' = 0$, then preferences are quasilinear. Finally, $u_1'' = u_2'' = 0$ refers to a situation in which both goods are income-neutral as in the Arrow-Pratt portfolio problem.[3]

Given Assumptions 2.1 and 2.2 and $\tilde{y} = \tilde{R}(I - x)$, (4) is rewritten as[4]

$$\max_{x \in (0,I)} \pi\varphi\left(u_1(x) + u_2(\overline{R}(I-x))\right) + (1-\pi)\varphi\left(u_1(x) + u_2(\underline{R}(I-x))\right). \quad (8)$$

Optimal consumption is defined by the first-order condition corresponding to (8), i.e.,

$$\begin{aligned}&\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right] \\ &\quad + (1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right] = 0 \quad (9)\end{aligned}$$

evaluated at $x = x^*$ so that the amount allocated to the risky good is $s^* \equiv I - x^*$. Here, for $R \in \{\overline{R}, \underline{R}\}$, $u_1'(x^*) - u_2'(R(I-x^*))R$ is the deterministic marginal utility of consumption for the sure good. From (9), the KM utility representation makes risk aversion explicit in optimal behavior under uncertainty. Specifically, since $\varphi'' < 0$, it follows that

$$0 < \varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) < \varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right). \quad (10)$$

Hence, risk aversion adds more weight to the deterministic marginal utility corresponding to the lowest rate of return.

In the next section, we use (9) to analyze the effect of changing income on optimal behavior. Under certainty, the income and substitution effects play an important role in determining the signs of the comparative statics. However, under uncertainty, the income and substitutions effects play an additional part in the comparative statics because they are also involved in signing the effect of risk aversion. This relationship between the income and substitution effects and the risk aversion effect was pointed out in MS. Specifically, MS characterizes the effect of changing risk aversion on the basis of the income and substitution effects under different sources of uncertainty. In other words, MS studies the *pure* risk aversion effect, i.e., the change in

[3] In the Arrow-Pratt portfolio problem, $V(x, \tilde{y}) = \varphi(x + \tilde{y})$.

[4] Note that, in this formulation, $V(x, \tilde{y}) = \varphi(u_1(x) + u_2(\tilde{y}))$ cannot be additive.

both the sure good and the risky good due to a change in risk aversion. In this paper, for the case of a random price of the risky good (i.e., the rate of return on the risky good), we show that the effect of changing risk aversion is part of the effect of changing income. Note that, for the pure risk aversion effect, income remains constant so that the change in the risky good offsets the change in the sure good. In the problem of changing income studied in this paper, the pure risk aversion effect must be modified to take account of the change in income.

Before proceeding, note that the income and substitution effects (related to a deterministic change in the rate of return) order the deterministic marginal utility of consumption for the sure good. To simplify the discussion, we hereafter refer to income and substitution effects without mentioning that these effects are related to a deterministic change in the rate of return. Formally,[5]

Remark 2.3. $u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R} > 0 > u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}$ *if and only if the income effect is stronger than the substitution effect.*[6]

[5]Let $MU(x, R; I) \equiv u_1'(x) - u_2'(R(I-x))R$ so that $\partial MU(x, R; I)/\partial R|_{x=x^*} = -u_2''(R(I-x^*))R^2 - u_2'(R(I-x^*))$ where $-u_2''(R(I-x^*))R^2 > 0$ and $-u_2'(R(I-x^*)) < 0$ are proportional to and of the same sign as the income effect and the substitution effect, respectively, related to a deterministic change in R.

[6]Note that we implicitly ignore the case in which the income and substitution effects cancel each other since, in this case, uncertainty has no effect on optimal behavior and risk aversion is thus irrelevant.

3 Comparative Analysis

We study, in this section, the effect of changing income on the sure good x^*, as well as the risky good through the amount saved, i.e., $s^* \equiv I - x^*$.[7] From (9), a change in income affects optimal behavior through the influence on risk aversion as well as the marginal utility of consumption. We proceed in several steps. We first decompose the effect of a change in income for the sure good and the risky good. We show that the effect of changing income depends on the effect of changing risk aversion. We then study the effect of changing income under different assumptions regarding attitudes toward risk and tastes. Finally, we show that in general there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965).

Decomposition of the Effect of Changing Income. We begin by decomposing the effect of changing income on the sure good. Propositions 3.1 states that the effect of changing income is determined by the sum of two terms, i.e., $\mathcal{MIE}_{x^*} + \mathcal{HE}_{x^*}$. The term $\mathcal{MIE}_{x^*}$ corresponds to the *modified income effect* on the sure good, i.e., the income effect modified by uncertainty. It has the same characteristics as the deterministic income effect, however it is modified to take account of the fact that the rate of return is random. The term $\mathcal{HE}_{x^*}$ is the *hybrid* effect that contains the effect of a change in risk aversion modified by the change in income. In other words, $\mathcal{MIE}_{x^*}$ captures the effect of changing income on the sure good through uncertainty whereas $\mathcal{HE}_{x^*}$ captures the effect of changing risk aversion due to changes in income. As noted, the symbol $\stackrel{S}{=}$ means *of the same sign as.*

[7] Specifically, a change in income changes the amount allocated to the risky good (i.e., savings), which induces a change in the distribution of the risky good. The effect of changing income on the risky good simply refers to the effect of changing income on savings.

Proposition 3.1. *From* (9),

$$\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathcal{MIE}_{x^*} + \mathcal{HE}_{x^*} \tag{11}$$

where

$$\begin{aligned}\mathcal{MIE}_{x^*} \equiv & -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) u_2''(\overline{R}(I-x^*))\overline{R}^2 \\ & -(1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) u_2''(\underline{R}(I-x^*))\underline{R}^2\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{HE}_{x^*} \equiv & -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right] \\ & \cdot \left(\frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)} u_2'(\overline{R}(I-x^*))\overline{R}\right. \\ & \left. - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)} u_2'(\underline{R}(I-x^*))\underline{R}\right).\end{aligned} \tag{13}$$

Proof. See Appendix B. □

Proposition 3.2 complements Proposition 3.1 by decomposing the effect of increasing income on s^*. As in the case of the sure good, the effect of changing income on the risky good is determined by both a modified income effect and a hybrid effect. However, these effects are different, i.e., $\mathcal{MIE}_{x^*} \neq \mathcal{MIE}_{s^*}$ and $\mathcal{HE}_{x^*} \neq \mathcal{HE}_{s^*}$.

Proposition 3.2. *From* (9),

$$\frac{\partial s^*}{\partial I} \stackrel{S}{=} \mathcal{MIE}_{s^*} + \mathcal{HE}_{s^*} \tag{14}$$

where

$$\begin{aligned}\mathcal{MIE}_{s^*} \equiv & -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) u_1''(x^*) \\ & -(1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) u_1''(x^*)\end{aligned} \tag{15}$$

and

$$\mathcal{HE}_{s^*} \equiv \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) u_1'(x^*) \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]$$
$$\cdot \left(\frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} \right). \tag{16}$$

Proof. See Appendix B. □

Two comments are in order for the case of additive preferences. First, from (12) and (15), $\mathcal{MIE}_{x^*} \geq 0$ and $\mathcal{MIE}_{s^*} \geq 0$. Whether the modified income effects are zero or strictly positive depends on the normality of the goods. Specifically, $\mathcal{MIE}_{x^*} > 0$ when the sure good is normal and $\mathcal{MIE}_{s^*} > 0$ when the risky good is normal. Hence, the signs of the modified income effects are independent of changes in risk aversion, and depend solely on ordinal preferences, i.e., tastes. Second, from (13) and (16), the signs of $\mathcal{HE}_{x^*}$ and $\mathcal{HE}_{s^*}$ depend on both attitudes toward risk and tastes. In other words, the hybrid effects combine the effect of risk aversion with changes in income on risk aversion. Although the functional forms of the hybrid effects differ between the two goods, both $\mathcal{HE}_{x^*}$ and $\mathcal{HE}_{s^*}$ are due to risk aversion, which is made explicit in our formulation.

Note that these hybrid effects are related to the pure risk aversion effect contained in MS. Indeed, in MS, the pure risk aversion effect is shown to depend on the income and substitution effects, i.e., the sign of $u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}$. Formally, let $a > 0$ be a coefficient of risk aversion such that an increase in a implies an increase in risk aversion. That is, for any z, $\partial\left(-\varphi''(z)/\varphi'(z)\right)/\partial a > 0$. Hence, from MS,[8]

$$\frac{\partial x^*}{\partial a} \stackrel{S}{=} -\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]. \tag{17}$$

However, with changes in income, the hybrid effect contains both changes in risk aversion and changes in income. In other words, the effect of changing

[8]Since there is no increase in income with the pure risk aversion effect, it follows that $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$.

income depends on the pure risk aversion effect through the hybrid effects. However, the hybrid effects are not completely analogous to the pure risk aversion effects because they are modified to take account of the changes in income, i.e., the changes in the level of utility, and thus alters the impact of the *pure* risk aversion effect.

To see this, rewrite the hybrid effects. For the sure good,

$$\mathcal{HE}_{x^*} \overset{S}{=} \frac{\partial x^*}{\partial a} \cdot \left(\frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} u_2'(\overline{R}(I - x^*))\overline{R} - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} u_2'(\underline{R}(I - x^*))\underline{R} \right) \tag{18}$$

where $\frac{\partial x^*}{\partial a}$ is defined by (17). For a change in income, the terms $u_2'(\overline{R}(I - x^*))\overline{R}$ and $u_2'(\underline{R}(I - x^*))\underline{R}$ representing changes in the marginal rates of substitution are weights for the change in risk aversion due to income. These terms combined determines the strength of the pure risk aversion effect for the hybrid effect. In other words, as income changes, not only does risk aversion change but tastes are also distorted so that the pure risk aversion effect is altered. The terms in parenthesis in (18) take account of the influence of both changes in risk aversion and changes in tastes on the pure risk aversion effect. Hence, the sign of the hybrid effect depends on the interaction between risk aversion, income and substitution effects. From (18), the income and substitution effects play a dual role in determining the sign of the effect of changing income. First, it orders the marginal utilities explicit in (18) which are used as weights for the effect of income on risk aversion. Here, the weights depend on the different levels of utility consistent with different levels of income. Second, it determines the sign of the pure risk aversion effect in (17) embedded in (18).

For the risky good, the income and substitution effects influence the sign of $\mathcal{HE}_{s^*}$ (and thus $\partial s^*/\partial I$) only through the risk aversion effect. As in (18), the effect of changing risk aversion influences the effect of changing income in a multiplicative way. In that case, the hybrid effect is the product of the effect of risk aversion and the effect of income on risk aversion without additional weights. That is, (15) is equivalent to

$$\mathcal{HE}_{s^*} \overset{S}{=} -\frac{\partial x^*}{\partial a}\cdot\left(\frac{-\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)} - \frac{-\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}\right) \tag{19}$$

where $\frac{\partial x^*}{\partial a}$ is defined by (17). Note finally that the hybrid effects in (18) and (19) can also be rewritten in terms of the effect of risk aversion on the risky good since $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$.[9]

Using Propositions 3.1 and 3.2 for the case of additive preferences, we now proceed to determine the direction of a change in income on optimal behavior under uncertainty. We then show that in general there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965).

Direction of a Change in Income. We first consider the case of constant absolute risk aversion and then analyze the case of decreasing absolute risk aversion. For each case, we discuss four situations. We begin with the case in which both goods are income-neutral as in the Arrow-Pratt portfolio problem. We then continue with quasilinear preferences and finish with both goods being normal.

Suppose that risk preferences exhibit constant absolute risk aversion. First, Proposition 3.3 shows that the hybrid risk aversion effect is different for the sure good and the risky good.[10] Proposition 3.3 states that under constant absolute risk aversion, the hybrid effect is present for the sure good but absent for the risky good. Specifically, the hybrid effect for the sure good is strictly positive due to the presence of risk aversion (i.e., $\varphi'' < 0$). In other words, the effect of changing income for the sure good in this case dominates the effect of risk aversion, and thus the amount of the sure good increases. However, the hybrid effect for the risky good is zero because, under constant absolute risk-aversion, an increase in income has no effect on the individual's

[9]Note also that the term representing changes in tastes as income changes seems to disappear. However, it can be seen from (18) to be the term $u_2'(x^*)$, which, due to the representation of preferences by additive utility, factors out.

[10]Note that for the case of pure risk aversion from MS, the effect of a change in risk aversion is the same (except for the signs) for both the sure good and the risky good.

risk aversion.

Proposition 3.3. *Suppose that risk preferences exhibit constant absolute risk-aversion. i.e.,*

$$-\frac{\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)} = -\frac{\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}. \tag{20}$$

Then,

1. *From* (13),

$$\begin{aligned}\mathcal{HE}_{x^*} \equiv\ & -\pi\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)\left[u_1'(x^*)-u_2'(\overline{R}(I-x^*))\overline{R}\right]^2 \\ & -(1-\pi)\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)\left[u_1'(x^*)-u_2'(\underline{R}(I-x^*))\underline{R}\right]^2 > 0.\end{aligned} \tag{21}$$

2. *From* (16), $\mathcal{HE}_{s^*} = 0$.

Under constant absolute risk aversion, we now consider differences regarding the normality of the goods. Proposition 3.4 states that with constant absolute risk aversion, when the risky good is income-neutral, a change in income is entirely allocated to the sure good, regardless of whether the sure good is normal (i.e., $u_2'' = 0$) or income-neutral (i.e., $u_2'' = 0$). In particular, under constant absolute risk aversion, a change in income does not affect risk aversion. Hence, since the risky good is income-neutral, the individual has no incentive to increase the amount of the risky good.

Proposition 3.4. *Suppose that risk preferences exhibit constant absolute risk aversion and that the risky good is income-neutral. Then,*

$$\frac{\partial x^*}{\partial I} = 1, \tag{22}$$

$$\frac{\partial s^*}{\partial I} = 0. \tag{23}$$

Proof. Suppose that the risky good is income neutral, i.e., $u_1'' = 0$. Then, from (14), (15), (16), and (20), $\frac{\partial s^*}{\partial I} = 0$. Since $s^* \equiv I - x^*$, it follows that $\frac{\partial x^*}{\partial I} = 1$. □

Consider next quasilinear preferences with the sure good income-neutral and the risky good normal. In that case, an increase in income induces an increase in both goods. Since the risky good is normal, the amount of the risky good increases because the modified income effect is strictly positive. As in Proposition 3.4, the sure good increases through the hybrid effect, although not as much since part of the new income is used for the risky good. While consumption of both goods increase with an increase in income, the reason for the increases are different. Indeed, the sure good increases because of risk aversion and the change in income through the hybrid effect whereas the risky good increases only because of the change in income.

Proposition 3.5. *Suppose that risk preferences exhibit constant absolute risk aversion. If the sure good is income-neutral and the risky good is normal, then*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due only to the hybrid effect, i.e.,* $\mathcal{MIE}_{x^*} = 0, \mathcal{HE}_{x^*} > 0$.
2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due only to the modified income effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} = 0$.

Finally, Proposition 3.6 states that when both goods are normal, an increase in income increases both the sure good and the risky good. Going from an income-neutral sure good to a normal sure good amplifies the positive effect of increasing income on the sure good.

Proposition 3.6. *Suppose that risk preferences exhibit constant absolute risk aversion. If both goods are normal, then*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due to the modified income and hybrid effects, i.e.,* $\mathcal{MIE}_{x^*} > 0, \mathcal{HE}_{x^*} > 0$.
2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due only to the modified income effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} = 0$.

Next, suppose that risk preferences exhibit decreasing absolute risk aversion. Proposition 3.7 states that under decreasing absolute risk aversion, the hybrid effect for the sure good is strictly positive whereas the hybrid effect for the risky good is positive or negative depending on the income and substitution effects.

Proposition 3.7. *Suppose that risk preferences exhibit decreasing risk aversion, i.e.,*

$$-\frac{\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)} < -\frac{\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}. \tag{24}$$

Then,

1. *From* (13), $\mathcal{HE}_{x^*} > 0$.
2. *From* (16), $\mathcal{HE}_{s^*} > 0$ *if and only if the substitution effect is stronger than the income effect.*

Proof. See Appendix B. □

We now consider decreasing absolute risk aversion with different assumptions regarding the normality of the goods. Proposition 3.8 states that when both goods are income-neutral, then an increase in income induces the individual to increase both the sure good and the risky good. To understand this, note that from MS, due to the substitution effect when the price of the risky good is random, a reduction in risk aversion induces the individual to decrease the sure good and increase the risky good. Under decreasing absolute risk aversion, an increase in income also induces a reduction in risk aversion, which then implies an increase in the risky good through the hybrid effect. However, unlike the effect of increasing risk aversion, the level of income has increased, which amplifies the increase in the consumption of the sure good.

Proposition 3.8. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that both goods are income-neutral. Then,*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due only to the hybrid effect,* $\mathcal{MIE}_{x^*} = 0, \mathcal{HE}_{x^*} > 0$.
2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due only to the hybrid effect,* $\mathcal{MIE}_{s^*} = 0, \mathcal{HE}_{s^*} > 0$.

Suppose that preferences are quasilinear with the sure good income-neutral (i.e., $u_2'' = 0$) and the risky good normal (i.e., $u_1'' < 0$). As in Proposition 3.8, the positive effect of increasing income on the sure good is due to the hybrid effect. For the risky good, the hybrid effect is positive due to the substitution effect since there is no income effect (i.e., $u_2'' = 0$). Compared to a income-neutral risky good, the positive effect of increasing income on the risky good is accentuated now that the risky good is normal. Indeed, for the risky good, both the modified income effect and the hybrid effect push in the direction of more consumption of the risky good.

Proposition 3.9. *Suppose that risk preferences exhibit decreasing absolute risk and that the sure good is income-neutral and the risky good is normal. Then,*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due only to the hybrid effect, i.e.,* $\mathcal{MIE}_{x^*} = 0, \mathcal{HE}_{x^*} > 0$.
2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due to the modified income and hybrid effects, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} > 0$.

Suppose next that preferences are quasilinear but now the sure good is normal (i.e., $u_2'' < 0$) and the risky good is income-neutral (i.e., $u_1'' = 0$). Here, the normality of the sure good makes the sign of the hybrid effect for the risky good ambiguous through the effect of increasing risk aversion. When the substitution effect is stronger than the income effect, the new income is allocated between the two goods. However, when the income effect is stronger than the substitution effect, the hybrid effect for the risky good is negative, which reduces consumption for the risky good.

Proposition 3.10. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that the sure good is normal and the risky good is income-neutral.*

1. *If the substitution effect is stronger than the income effect, then* $\frac{\partial x^*}{\partial I}, \frac{\partial s^*}{\partial I} \in (0,1)$.
2. *If the income effect is stronger than the substitution effect, then* $\frac{\partial x^*}{\partial I} > 1$ *and* $\frac{\partial s^*}{\partial I} < 0$.

From Propositions 3.9 and 3.10, the normality of the good in quasilinear preferences has a profound effect on the comparative analysis. In the case of an income-neutral sure good, an increase in income is allocated to both sure and risky goods. The increase in the sure good is due to the effect of increasing income on risk aversion whereas the increase in the risky good is due to a pure income effect. In the case of an income-neutral risky good, the normality of the sure good makes it possible for the hybrid effect corresponding to the risky good to be negative. In that case, an increase in income induces an increase in the sure good (due to the presence of more income) as well as a reallocation from the risky good to the sure good due to the pure risk aversion effect contained in the hybrid effect for the risky good. In that case, $\frac{\partial x^*}{\partial I} > 1$.

Suppose finally that both goods are normal, (i.e., $u_1'', u_2'' < 0$). In that case, the effect of increasing income on the risky good is ambiguous. Here, the normality of the risky good (i.e., $u_1'' < 0$) makes the modified income effect positive so that the individual increases savings. At the same time, the fact that $u_2'' < 0$ makes it possible for the hybrid effect to reduce savings. This happens if the income effect is stronger than the substitution effect.

Proposition 3.11. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that both goods are normal. Then,* $\frac{\partial x^*}{\partial I} > 0$ *and*

1. *If the substitution effect is stronger than the income effect, then* $\frac{\partial s^*}{\partial I} > 0$ *due to the modified income and hybrid effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} > 0$.

2. *If the income effect is stronger than the substitution effect, then* $\frac{\partial s^*}{\partial I} < 0$ *if and only if* $\mathcal{MIE}_{s^*} + \mathcal{HE}_{s^*} < 0$.

On Equivalence. Finally, we show that in general there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965). Indeed, the equivalence depends on the assumption regarding tastes, i.e., whether goods are normal or income-neutral. It also depends on the fact that it is the price of the risky good that is random. Indeed, if income or the price of the sure good were random, the result would change.

We begin with the Arrow-Pratt portfolio problem in which both goods are income-neutral, i.e., $V(x, \tilde{y}) = \varphi(x + \tilde{y})$. For Arrow (1965), increasing income when risk preferences exhibit decreasing absolute risk aversion makes the individual less risk averse and thus the amount allocated to the risky good is increased while, for Pratt (1964), increasing risk aversion decreases the amount of the risky good, so that a decrease in risk aversion increases the risky good.

In the case of income-neutral goods, the modified income effect is absent and thus the effect of changing income is entirely linked to the effect of changing risk aversion through the hybrid effect, i.e., using (19) and $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$, decreasing absolute risk aversion implies that

$$\frac{\partial s^*}{\partial I} \overset{S}{=} \mathcal{HE}_{s^*} \tag{25}$$

$$\frac{\partial s^*}{\partial a} \overset{S}{=} -\mathcal{HE}_{s^*} \tag{26}$$

Moreover, since the sure good is income-neutral, the substitution effect is dominant so that $\mathcal{HE}_{s^*} < 0$. Proposition 3.12 follows immediately.

Proposition 3.12. *(Arrow-Pratt) Suppose that both goods are income-neutral. Then, the following two statements are* ***equivalent.***

1. *An decrease in risk aversion decreases the amount allocated to the risky good.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income decreases the amount allocated to the risky good.*

Suppose now that the risky good is normal and the sure good is income-neutral, i.e., $V(x, \tilde{y}) = \varphi(u_1(x) + \tilde{y})$. Hence, $\mathcal{MIE}_{s^*} > 0$ whereas $\mathcal{HE}_{s^*} < 0$ due to the substitution effect as in the case of two income-neutral goods. The equivalence does not hold since the positive modified income effect induces more consumption for the risky good. Specifically, if the positive modified income effect is stronger than the negative hybrid effect, then, under DARA, an increase in income induces more consumption for the risky good. On the other hand, a decrease in risk aversion induces less consumption for the risky good.

Proposition 3.13. *Suppose that the sure good is income-neutral and that the risky good is normal. Then, the following two statements are* ***not equivalent.***

1. *An decrease in risk aversion increases the amount allocated to the risky good if and only if the modified income effect is stronger than the hybrid effect.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income decreases the amount allocated to the risky good due to the substitution effect.*

Finally, suppose that both goods are normal. As in the previous case, the normality of the risky good implies that an increase in income induces more consumption of the risky good through the modified income effect (i.e., $\mathcal{MIE}_{s^*} > 0$). However, the normality of the sure good implies that an increase in income induces less consumption of the risky good through the hybrid effect (i.e., $\mathcal{HE}_{s^*} < 0$) if the income effect is stronger than the substitution effect. Note that the source of the nonequivalence is solely due to the normality of the risky good. Indeed, because the normality of the sure good affects the sign only through the hybrid effect which is directionally equivalent to the sign of the effect of risk aversion, the normality of the sure good reinforces the equivalence result in the case of two income-neutral goods while the normality of the risky good pulls in the opposite direction.

Proposition 3.14. *Suppose that both goods are normal. Then, the following two statements are* ***not equivalent.***

1. *A decrease in risk aversion decreases the amount allocated to the risky good if and only if the income effect is stronger than the substitution effect.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income increases the amount allocated to the risky good if and only if the modified income effect is stronger than than the hybrid effect.*

Hence, for the risky good, the equivalence is not general. Consider next the sure good. Proposition 3.15 states that for the sure good there is no equivalence between the Arrow result and the Pratt result. In the case of the sure good, an increase in income induces more consumption of the sure good. This is regardless of the normality of the goods since $\mathcal{MIE}_{x^*} \geq 0$ and under DARA $\mathcal{HE}_{x^*} > 0$. However, the effect of risk aversion depends on the income and substitution effects.

Proposition 3.15. *The following two statements are **not equivalent.***

1. *An decrease in risk aversion increases the sure good if and only if the income effect is stronger than the substitution effect.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income increases the sure good.*

4 Discussion

Having studied the effect of changing income on optimal behavior and highlighted the role of classical demand theory and risk aversion on comparative statics, we now discuss the appropriateness of an alternative approach to study risk aversion suggested by Selden (1978), which has been widely popularized through the parametric model of Epstein and Zin (1989) (EZ).

Specifically, in this section, we present the effects of changing income and changing risk aversion on optimal behavior under uncertainty using EZ preferences. We then compare these comparative statics results with the ones corresponding to the KM approach. In Appendix C, we explain why the Selden-EZ preferences cannot be used to isolate the effect of risk aversion.

As noted in Appendix C, the Selden-EZ approach uses the certainty equivalent to reflect risk aversion. Formally, given (44), for any gamble g on $(x, \tilde{y})$, the Selden-EZ utility function is $W_S(x, \tilde{y}) = u_1(x) + u_2(\mu(\tilde{y}, v_S))$, where $\mu(\tilde{y}, v_S) = v_S^{-1}(\mathbb{E}_{\tilde{y}} v_S(\tilde{y}))$ is the certainty equivalent. Here, $\mathbb{E}_{\tilde{y}}$ is the expectation operator with respect to $\tilde{y}$ and v_S is a strictly increasing and concave function, $v_S' > 0, v_S'' \leq 0$. In the Selden-EZ approach, a decrease in $\mu(\tilde{y}, v_S)$ due to a more concave v_S is used to mean that the individual is more risk averse.

Consider now the EZ parametric model. That is, suppose that $u_1(z) = u_2(z) = z^{1-\rho}$ and $v_S(z) = z^{1-\gamma}$, $\gamma \neq 1$, where γ represents the one-dimensional coefficient of risk aversion and the parameter ρ represents the intertemporal elasticity of substitution under certainty. Hence, given Assumption 2.1 and $\tilde{y} = \tilde{R}(I - x)$, the consumption-saving problem under EZ preferences is

$$\max_x x^{1-\rho} + \left(\pi \underline{R}^{1-\gamma}(I-x)^{1-\gamma} + (1-\pi)\overline{R}^{1-\gamma}(I-x)^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}}. \qquad (27)$$

From (27), optimal consumption under EZ preferences is[11]

$$x^{EZ} = \frac{I}{1 + \left(\pi \underline{R}^{1-\gamma} + (1-\pi)\overline{R}^{1-\gamma}\right)^{\frac{1-\rho}{(1-\gamma)\rho}}}, \tag{28}$$

$$s^{EZ} = \frac{\left(\pi \underline{R}^{1-\gamma} + (1-\pi)\overline{R}^{1-\gamma}\right)^{\frac{1-\rho}{(1-\gamma)\rho}} I}{1 + \left(\pi \underline{R}^{1-\gamma} + (1-\pi)\overline{R}^{1-\gamma}\right)^{\frac{1-\rho}{(1-\gamma)\rho}}}. \tag{29}$$

We now perform a comparative analysis on expressions (28) and (29). Remark 4.1 states that the effect of income on behavior under EZ preferences is to increase both the sure and the risky amount. For the case of a change in income, EZ preferences disregard the effect of an increase in income on risk aversion, which, under decreasing absolute risk aversion, makes the individual less risk averse.

Remark 4.1. *From* (28) *and* (29)*,* $\partial x^{EZ}/\partial I, \partial s^{EZ}/\partial I > 0$*.*

Remark 4.2 states that under EZ preferences an increase in the one-dimensional coefficient of risk aversion γ leads to an increase in the sure good and thus a decrease in the risky good. Indeed, from (28) and (29), an increase in γ decreases the term $\left(\pi \underline{R}^{1-\gamma} + (1-\pi)\overline{R}^{1-\gamma}\right)^{\frac{1-\rho}{(1-\gamma)\rho}}$. Hence, EZ preferences ignore the role played by the income and substitution effects as shown in MS in order to determine the effect of changing risk aversion.

Remark 4.2. *From* (28) *and* (29)*,* $\partial x^{EZ}/\partial \gamma > 0$ *and* $\partial s^{EZ}/\partial \gamma < 0$*.*

[11] Taking the first-order condition corresponding to (27) yields (28). Plugging (28) into $s^{EZ} \equiv I - x^{EZ}$ yields (28).

A KM Example

To show that attitudes toward risk and tastes are not separated, we recall the example stated in KM. Let $V^1(x, y)$ and $V^2(x, y)$ be two distinct utility functions yielding indifference curves of the type IC_1 and IC_2, respectively, as depicted in Figure 1. Let (x_A, y_A) and (x_B, y_B) be two distinct consumption bundles such that $V^1(x_A, y_A) > V^1(x_B, y_B)$ and $V^2(x_A, y_A) < V^2(x_B, y_B)$. Consider choosing between the sure outcome yielding (x_A, y_A) and a gamble yielding (x_A, y_A) with probability $\pi \in (0, 1]$ and (x_B, y_B) with probability $1 - \pi$.

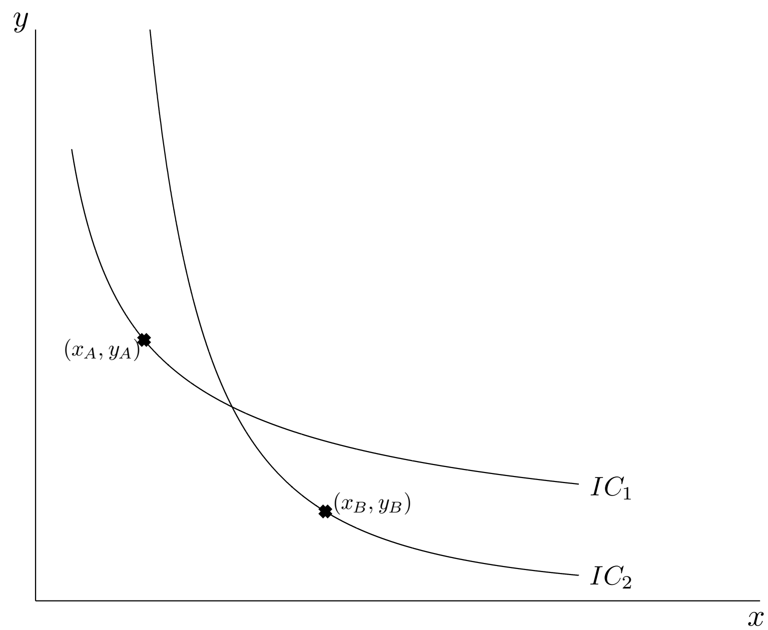

Figure 1: KM Example

Consistent with Figure 1, an individual with preferences $V^1(x, y)$ prefers the sure outcome, while an individual with preferences $V^2(x, y)$ prefers the gamble.[12] The individual with preferences $V^2(x, y)$ acts in a seemingly more

[12]In other words, $V^1(x_A, y_A) > \pi V^1(x_A, y_A) + (1 - \pi)V^1(x_B, y_B)$ and $V^2(x_A, y_A) < \pi V^2(x_A, y_A) + (1 - \pi)V^2(x_B, y_B)$.

risk-averse way than the individual with preferences $V^1(x, y)$, but is not more risk-averse. Rather, it is the composition of goods in the gamble that is preferred.

B Proofs

Proof of Propositions 3.1 and 3.2. From (9),

$$\frac{\partial x^*}{\partial I} = \frac{\Omega}{-\Delta} \tag{30}$$

and, since $s^* \equiv I - x^*$,

$$\frac{\partial s^*}{\partial I} = 1 - \frac{\Omega}{-\Delta}, \tag{31}$$

where

$$\begin{aligned}\Omega \equiv & \left(\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} u_2'(\overline{R}(I - x^*))\overline{R} \right. \\ & \left. - \frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} u_2'(\underline{R}(I - x^*))\underline{R} \right) \\ & \cdot \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] \\ & - \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) u_2''(\overline{R}(I - x^*))\overline{R}^2 \\ & - (1 - \pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) u_2''(\underline{R}(I - x^*))\underline{R}^2 \end{aligned} \tag{32}$$

and

$$\Delta \equiv \left(\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)} \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right] \right.$$
$$\left. - \frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)} \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right] \right)$$
$$\cdot \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right]$$
$$+ \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \left[u_1''(x^*) + u_2''(\overline{R}(I-x^*))\overline{R}^2\right]$$
$$+ (1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) \left[u_1''(x^*) + u_2''(\underline{R}(I-x^*))\underline{R}^2\right]. \tag{33}$$

Note that the second-order condition implies that $\Delta < 0$. Rearranging (30) and (31) yields the decomposition stated in Propositions 3.1 and 3.2.

Proof of Proposition 3.3. Let

$$\Delta = -\pi\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right]^2$$
$$- (1-\pi)\varphi''\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right]^2 \tag{34}$$

which is negative since $\varphi'' < 0$. Expression (34) can be rewritten as

$$\Delta = \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)} \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right]^2$$
$$+ \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)} (1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)$$
$$\cdot \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right]^2. \tag{35}$$

Multiplying both sides of (9) by $[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}]$ yields

$$(1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right]^2$$
$$= -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right] \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right]. \tag{36}$$

Plugging (36) into (35) yields

$$
\begin{aligned}
\Delta = & \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]^2 \\
& - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \\
& \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]\left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right], \qquad (37) \\
= & \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} \left(u_2'(\underline{R}(I - x^*))\underline{R} - u_2'(\overline{R}(I - x^*))\overline{R}\right) \qquad (38) \\
& \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right], \qquad (39) \\
= & \mathcal{HE}_{x^*}. \qquad (40)
\end{aligned}
$$

Proof of Proposition 3.7. Let

$$
\begin{aligned}
\Gamma = & \pi\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]^2 \\
& + (1 - \pi)\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]^2 < 0. \qquad (41)
\end{aligned}
$$

Using (9), we can rewrite the above as

$$
\begin{aligned}
\Gamma = & \left(\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] \right. \\
& \left. - \frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right] \right) \\
& \cdot \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] < 0 \qquad (42)
\end{aligned}
$$

so that

$$\begin{aligned}
&\pi\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)\left[u_1'(x^*)-u_2'(\overline{R}(I-x^*))\overline{R}\right]\\
&\cdot\left(\frac{-\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}u_2'(\overline{R}(I-x^*))\overline{R}\right.\\
&\left.-\frac{-\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}u_2'(\underline{R}(I-x^*))\underline{R}\right)\\
&<\underbrace{\left(\frac{\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}-\frac{\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}\right)}_{\leq 0}u_1'(x^*).
\end{aligned} \tag{43}$$

The left-hand side of the above equation is HE_{x^*} so that $HE_{x^*} > 0$ whether risk preferences exhibit CARA or DARA.

C Discussion

In this appendix, we discuss the appropriateness of an alternative approach to study risk aversion suggested by Selden (1978). We show that the Selden-EZ approach cannot be used to isolate the effect of risk aversion.

Two approaches have been suggested to disentangle tastes from risk aversion, and, thus, to analyze the effect of risk aversion on behavior. The first established by Kihlstrom and Mirman (1974) (henceforth, KM) considers the class of utilities that are concave transformations. Formally, let

$$U(x, y) = u_1(x) + u_2(y), \tag{44}$$

$u'_1, u'_2 > 0, u''_1, u''_2 \leq 0$ be the utility associated with the consumption profile $(x, y) \in \mathbb{R}^2_+$.[13] Given (44), for any gamble g on $(x, \tilde{y})$ in which x is the sure good and $\tilde{y}$ is the risky good, the KM utility function is

$$W_{KM}(x, \tilde{y}) = \mathbb{E}_{\tilde{y}} v_{KM} \left(u_1(x) + u_2(\tilde{y}) \right), \tag{45}$$

where $\mathbb{E}_{\tilde{y}}$ is the expectation operator with respect to $\tilde{y}$, and v_{KM} is a strictly increasing and concave function, $v'_{KM} > 0, v''_{KM} \leq 0$.[14]

As noted, the KM approach can be used to study the effect of risk aversion on behavior because concave transformations of the utility function alter the expected marginal rate of substitution in a way that is consistent with ordinal preferences. To see this, consider the two gambles,

$$g_A \equiv \Big(\pi \circ (x_A, \underline{y}_A), (1 - \pi) \circ (x_A, \overline{y}_A) \Big), \tag{46}$$

$$g_B \equiv \Big(\pi \circ (x_B, \underline{y}_B), (1 - \pi) \circ (x_B, \overline{y}_B) \Big), \tag{47}$$

where, for $i = A, B$, $\underline{y}_i < \overline{y}_i$ and $\pi \in [0, 1]$ is the probability of receiving $(x_i, \underline{y}_i)$ in gamble i. We make two further restrictions. First, the gambles are not on the same vertical lines, i.e., $x_A < x_B$. Second, $\underline{y}_A > \underline{y}_B$ and $\overline{y}_A > \overline{y}_B$,

[13]We consider additive utility functions only for clarity. The discussion applies to more general utility functions.

[14]Note that, in this formulation, W_{KM} cannot be additive.

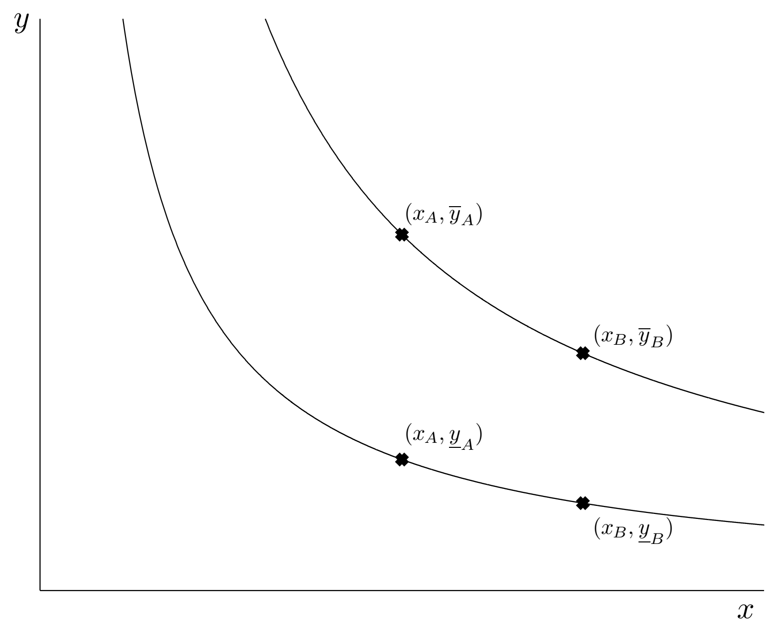

Figure 2: Case 1

i.e., $\tilde{y}_A$ first-order stochastically dominates $\tilde{y}_B$.

Suppose that ordinal preferences over the bundles are as depicted in Figure 2, i.e.,

$$u_1(x_A) + u_2(\underline{y}_A) = u_1(x_B) + u_2(\underline{y}_B), \tag{48}$$

$$u_1(x_A) + u_2(\overline{y}_A) = u_1(x_B) + u_2(\overline{y}_B). \tag{49}$$

Proposition C.1 states that a KM concave transformation does not alter the ordering of these two gambles. Indeed, from (46), (47), (48), and (49), the KM utilities for the two gambles are identical, i.e.,

$$\begin{aligned} &\pi v_{KM}\left(u_1(x_A) + u_2(\underline{y}_A)\right) + (1-\pi) v_{KM}\left(u_1(x_A) + u_2(\overline{y}_A)\right) \\ &= \pi v_{KM}\left(u_1(x_B) + u_2(\underline{y}_B)\right) + (1-\pi) v_{KM}\left(u_1(x_B) + u_2(\overline{y}_B)\right). \end{aligned} \tag{50}$$

Formally,

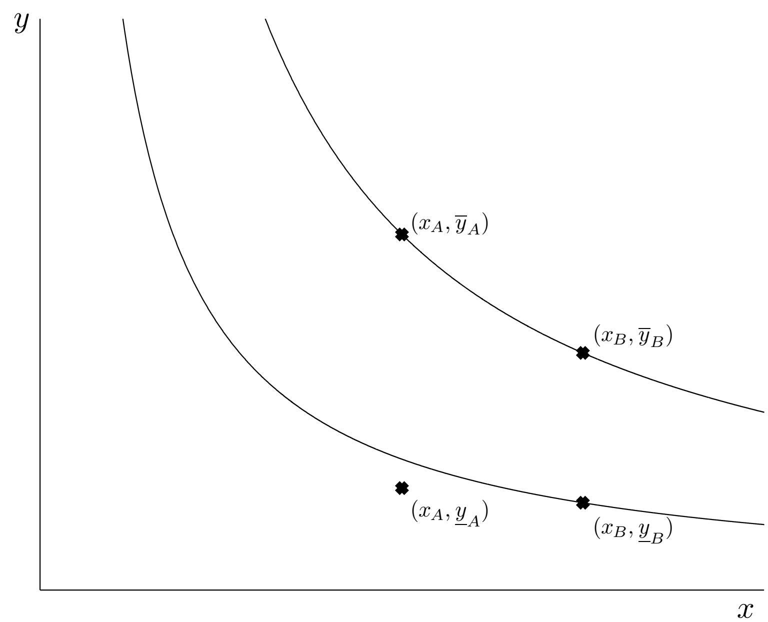

Figure 3: Case 2

Proposition C.1. *Suppose* (48) *and* (49) *hold. Under KM preferences, for any concave transformation* v_{KM}*, an individual is indifferent between gamble* A *and gamble* B*.*

Suppose next that ordinal preferences over the bundles are as depicted in Figure 3, i.e.,

$$u_1(x_A) + u_2(\underline{y}_A) < u_1(x_B) + u_2(\underline{y}_B), \tag{51}$$

$$u_1(x_A) + u_2(\overline{y}_A) = u_1(x_B) + u_2(\overline{y}_B). \tag{52}$$

That is, in terms of utility levels, gamble A is strictly worse than gamble B. Proposition C.2 states that, regardless of the concave transformation v_{KM}, gamble B is always strictly preferred to gamble A. Indeed, for $\pi \in [0, 1)$,

$W_{KM}(x_A, \tilde{y}_A) < W_{KM}(x_B, \tilde{y}_B)$.[15] Formally,

Proposition C.2. *Suppose* (51) *and* (52) *hold, and* $\pi \in [0, 1)$*. Under KM preferences, for any concave transformation* v_{KM}*,* $g_B \succ g_A$*.*

The second approach suggested by Selden (1978) uses the certainty equivalent as a measure of risk aversion. Formally, given (44), for any gamble g on $(x, \tilde{y})$, the Selden-EZ utility function is

$$W_S(x, \tilde{y}) = u_1(x) + u_2(\mu(\tilde{y}, v_S)), \tag{54}$$

where

$$\mu(\tilde{y}, v_S) = v_S^{-1}(\mathbb{E}_{\tilde{y}} v_S(\tilde{y})) \tag{55}$$

is the certainty equivalent. Here, $\mathbb{E}_{\tilde{y}}$ is the expectation operator with respect to $\tilde{y}$ and v_S is a strictly increasing and concave function, $v_S' > 0, v_S'' \leq 0$. In the Selden-EZ approach, a decrease in $\mu(\tilde{y}, v_S)$ due to a more concave v_S is used to mean that the agent is more risk averse. The basis for this approach is the certainty equivalence of the one dimensional Arrow-Pratt theory of risk-aversion. However, while there is an equivalence between a positive risk premium (or a certainty equivalent) and a concave transformation of the utility function in the one-dimensional case, this is not true in the multidimensional case.

In fact, unlike KM preferences, Selden-EZ preferences distort the expected marginal rate of substitution in a way that yields choices that are inconsistent with ordinal preferences. Selden-EZ preferences do not fall into the same category as the KM preferences because Selden-EZ preferences do not follow from a concave transformation. Indeed, a change in the concavity of v_S is equivalent to a concave transformation on the second utility function u_2. This partial concave transformation in Selden-EZ preferences is the reason

[15] From (46), (47), (51), and (52), for $\pi \in [0, 1)$,

$$\begin{aligned} &\pi v_{KM}\left(u_1(x_A) + u_2(\underline{y}_A)\right) + (1-\pi) v_{KM}\left(u_1(x_A) + u_2(\overline{y}_A)\right) \\ &< \pi v_{KM}\left(u_1(x_B) + u_2(\underline{y}_B)\right) + (1-\pi) v_{KM}\left(u_1(x_B) + u_2(\overline{y}_B)\right). \end{aligned} \tag{53}$$

that the Selden-EZ utility representation conflates tastes with risk aversion. Moreover, unlike the KM measure of risk aversion, the Selden-EZ measure of risk aversion can only be studied when there is a specific gamble. Indeed, without a gamble, preferences revert to the original deterministic preferences, so that v_S is only relevant with respect to a specific gamble.

The problems with the choice of gambles in the Selden-EZ approach is subtler than in the KM example of Figure 1. The KM example does not apply to Selden-EZ preferences because Selden-EZ preferences represent the same deterministic preferences, i.e., the same indifference curves. However, Selden-EZ preferences do not represent consistent preferences over gambles since changes in the concavity of v_S also changes tastes for gambles. In order to show this inconsistency, we need a more subtle example using the fact that deterministic preferences are the same. In fact, we can use the gambles defined by (46) and (47) to show that an inconsistency arises. Suppose that the ordinal preferences over the bundles are as depicted in Figure 2. In contrast to Proposition C.1, Proposition C.3 states that the Selden-EZ approach alters the ordering of these two gambles. In fact, gamble A can be preferred to gamble B because the expected return on the risky good $\tilde{y}_A$ is strictly greater than the expected return on the risky good $\tilde{y}_B$. This is important because it shows that Selden-EZ preferences disregard tastes in favor of first-order stochastic dominance on the value of outcomes in the risky good. Moreover, the fact that Selden-EZ preferences chooses gamble A is unrelated to the riskiness of the values of the risky good. In fact, from Figure 2, even though gamble A is preferred, $\overline{y}_A - \underline{y}_A > \overline{y}_B - \underline{y}_B$.

Proposition C.3. *Suppose* (48) *and* (49) *hold. Under Selden-EZ preferences, gamble A can be strictly preferred to gamble B.*

Proof. Let

$$f_A(\pi) = u_1(x_A) + u_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right), \tag{56}$$

$$f_B(\pi) = u_1(x_B) + u_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right), \tag{57}$$

be the Selden-EZ utilities as a function of π. From (48), (49), (56), and (57),

$f_A(0) = f_B(0)$ and $f_A(1) = f_B(1)$. Moreover,

$$f'_A(\pi) = \frac{u'_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right)\left(v_S(\underline{y}_A) - v_S(\overline{y}_A)\right)}{v'_S\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right)} < 0, \tag{58}$$

$$f'_B(\pi) = \frac{u'_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right)\left(v_S(\underline{y}_B) - v_S(\overline{y}_B)\right)}{v'_S\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right)} < 0. \tag{59}$$

Evaluating (58) and (59) at $\pi = 1$ yields

$$f'_A(\pi)|_{\pi=1} = \frac{u'_2(\underline{y}_A)\left(v_S(\underline{y}_A) - v_S(\overline{y}_A)\right)}{v'_S(\underline{y}_A)} < 0, \tag{60}$$

$$f'_B(\pi)|_{\pi=1} = \frac{u'_2(\underline{y}_B)\left(v_S(\underline{y}_B) - v_S(\overline{y}_B)\right)}{v'_S(\underline{y}_B)} < 0. \tag{61}$$

When $\underline{y}_A$ is close to $\underline{y}_B$,

$$f'_A(\pi)|_{\pi=1} < f'_B(\pi)|_{\pi=1} < 0, \tag{62}$$

so that for some $\pi \in (0,1)$ close to $\pi = 1$, $f_A(\pi)|_{\pi\approx 1} > f_B(\pi)|_{\pi\approx 1}$, i.e., gamble A is strictly preferred to gamble B. □

Suppose next that the ordinal preferences over the bundles are as depicted in Figure 3. In contrast to Proposition C.2, Proposition C.4 states that the ordering over the two gambles can be inconsistent with ordinal preferences. That is, gamble A which is strictly worse (in terms of utility outcomes) than gamble B can be chosen under the Selden-EZ approach. Moreover, the fact that Selden-EZ preferences chooses gamble A is unrelated to the riskiness of the utilities corresponding to the values of the risky good. In fact, from Figure 3, even though $u_2(\overline{y}_A) - u_2(\underline{y}_A) > u_2(\overline{y}_B) - u_2(\underline{y}_B)$, gamble A is preferred. It should also be noted that, for given $\pi \in (0,1)$ for which gamble A is strictly preferred to gamble B, increasing the concavity of v_S eventually leads to a reversal of the ordering of the gambles, i.e., for very concave v_S, gamble B is preferred to gamble A. Indeed, as v_S becomes more concave, the certainty equivalent tends toward the lowest utility, and, from (51), the individual no longer neglects the issue of tastes and jumps back to gamble B.

Proposition C.4. *Suppose* (51) *and* (52) *hold. Under Selden-EZ preferences, gamble A can be preferred to gamble B.*

Proof. From (51), (52), (56), and (57), $f_A(0) = f_B(0)$ and $f_A(1) < f_B(1)$. Moreover, evaluating (58) and (59) at $\pi = 0$ yields

$$f'_A(\pi)|_{\pi=0} = \frac{u'_2(\overline{y}_A)\left(v_S(\underline{y}_A) - v_S(\overline{y}_A)\right)}{v'_S(\overline{y}_A)} < 0, \tag{63}$$

$$f'_B(\pi)|_{\pi=0} = \frac{u'_2(\overline{y}_B)\left(v_S(\underline{y}_B) - v_S(\overline{y}_B)\right)}{v'_S(\overline{y}_B)} < 0. \tag{64}$$

When u and v_S are such that both[16]

$$\frac{u'_2(\overline{y}_A)}{v'_S(\overline{y}_A)} < \frac{u'_2(\overline{y}_B)}{v'_S(\overline{y}_B)} \tag{66}$$

and

$$v_S(\underline{y}_A) - v_S(\overline{y}_A) < v_S(\underline{y}_B) - v_S(\overline{y}_B), \tag{67}$$

then

$$0 > f'_A(\pi)|_{\pi=0} > f'_B(\pi)|_{\pi=0}, \tag{68}$$

so that for some $\pi \in (0, 1)$ close to $\pi = 0$, $f_A(\pi)|_{\pi\approx 0} > f_B(\pi)|_{\pi\approx 0}$, i.e., gamble A is strictly preferred to gamble B. □

Propositions C.3 and C.4 show that the certainty equivalent in the multi-dimensional case cannot be compared in a meaningful way when considering gambles that are on different vertical lines, i.e., $g_i \equiv (\pi \circ (x_i, y_i), (1-\pi) \circ$

[16] This occurs when, for all z,

$$\frac{u''_2(z)}{u'_2(z)} < \frac{v''_S(z)}{v'_S(z)}. \tag{65}$$

$(x_i', y_i'))$, $x_i \neq x_i', y_i \neq y_i'$.[17] In fact, implicit in the comparison across different vertical lines are the tastes or preferences corresponding to the points on these two different vertical lines. Changing the concavity of v_S in Selden-EZ preferences thus conflate risk aversion and tastes.

As noted, the inconsistency regarding ordinal preferences occurs because the expected marginal rate of substitution is distorted by the Selden-EZ approach. To see this, we now present the expected marginal rate of substitution under both KM and Selden-EZ preferences. Consider the gamble

$$g \equiv (\pi \circ (x, y+\varepsilon), (1-\pi) \circ (x, y-\varepsilon)) \tag{69}$$

for $\pi \in (0,1)$ and $y > \varepsilon \geq 0$.

Using (45), the KM utility function is

$$W_{KM}(x, \tilde{y}) = \pi v_{KM}(u_1(x)+u_2(y+\varepsilon)) + (1-\pi) v_{KM}(u_1(x)+u_2(y-\varepsilon)), \tag{70}$$

where $v_{KM}' > 0, v_{KM}'' \leq 0$. Here, the expected marginal rate of substitution is

$$\frac{\partial y}{\partial x} = -\frac{u_1'(x)}{\rho(v_{KM})u_2'(y+\varepsilon) + (1-\rho(v_{KM}))u_2'(y-\varepsilon)}, \tag{71}$$

where

$$\rho(v_{KM}) \equiv \frac{\pi v_{KM}'(u_1(x)+u_2(y+\varepsilon))}{\pi v_{KM}'(u_1(x)+u_2(y+\varepsilon)) + (1-\pi) v_{KM}'(u_1(x)+u_2(y-\varepsilon))}. \tag{72}$$

Note that, for a given gamble, since the two values of $\tilde{y}$ occur on separate indifference curves, the expected marginal rate of substitution is a convex combination of the marginal rates of substitution under certainty. Using (54), the Selden-EZ utility function is rewritten as

$$W_S(x, y+\tilde{\varepsilon}) = u_1(x) + u_2(\mu(y+\tilde{\varepsilon}, v_S)), \tag{73}$$

[17] Only gambles that have their same first argument (i.e., gambles on the same vertical line) can be compared, e.g., $g_i \equiv (\pi \circ (x, y_i), (1-\pi) \circ (x, y_i'))$, $y_i \neq y_i'$ using the certainty equivalent approach. That is, it is only when restricting attention to gambles on a vertical line that an increase in the concavity of v_S (yielding a decrease in the certainty equivalent) is related to risk aversion.

where

$$\mu(y+\tilde{\varepsilon},v_S)=v_S^{-1}(\pi v_S(y+\varepsilon)+(1-\pi)v_S(y-\varepsilon)) \tag{74}$$

is the certainty equivalent. Here, the expected marginal rate of substitution is

$$\frac{\partial y}{\partial x}=-\frac{u_1'(x)}{u_2'(\mu(y+\tilde{\varepsilon},v_S))\frac{\partial \mu(y+\tilde{\varepsilon},v_S)}{\partial y}}<0. \tag{75}$$

On the one hand, from (71), the KM approach affects the weights on the marginal utilities of the second argument, without affecting the values on the marginal utilities themselves. On the other hand, from (75), with the Selden-EZ approach, the marginal utility of the second argument is evaluated at the certainty equivalent and is distorted by the derivative of the certainty equivalent with respect to the outcome of y. This distortion has the effect of changing the ordering preferences over the gambles.

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