Centre Interuniversitaire sur le Risque, les Politiques Économiques et l'Emploi

Cahier de recherche/Working Paper **09-38**

Share the Gain, Share the Pain? Almost Transferable Utility, Changes in Production Possibilities, and Bargaining Solutions

Elisabeth Gugl

Justin Leroux

Août/August 2009

Gugl : Department of Economics, University of Victoria, P.O. Box 1700, STN CSC, Victoria, B.C., Canada V8W 2Y2; Tel.: (+1) 250 721-8538
egugl@uvic.ca
Leroux: HEC Montréal and CIRPÉE, 3000 chemin de la Côte-Ste-Catherine, Montréal, QC, Canada H3T 2A7; Tel. : (+1) 514 340-6864
justin.leroux@hec.ca

Gugl thanks Nick Baigent, Craig Brett, Philip Curry, Peter Kennedy, Shelly Lundberg, Hervé Moulin, David Scoones, Paul Schure and Linda Welling for their helpful comments on earlier versions of this paper. We also thank our discussant at the WEAI 84th conference in Vancouver, Jamie Woods for his comments on the current version.

This research was made possible in part thanks to support from FQRSC, which is gratefully acknowledged.

Abstract:
Consider an n-person economy in which efficiency is independent of distribution but the cardinal properties of the agents' utility functions precludes transferable utility (a property we call "Almost TU"). We show that Almost TU is a necessary and sufficient condition for all agents to either benefit jointly or suffer jointly with any change in production possibilities under well-behaved generalized utilitarian bargaining solutions (of which the Nash Bargaining and the utilitarian solutions are special cases). We apply the result to household decisionmaking in the contex of the Rotten Kid Theorem and in evaluating a change in family taxation.

Keywords: Axiomatic bargaining, solidarity, transferable utility, familyT-taxation, Rotten Kid Theorem

JEL Classification: C71, D63, D13

1 Introduction

Solidarity is a crucially important consideration whenever individuals cooperatively decide on how to allocate resources among themselves: an agreement is unlikely to be reached if some are hurt while others benefit in the event of a foreseeable shock in the available resources. From a normative standpoint, solidarity is also a chief concern when making policy recommendations, as illustrated by the following slightly modified example found in Nash (1950). Suppose Jack and Bill are siblings and have to share the following items: a book, a whip, a ball, a bat, a box, a pen, a toy, a knife, and a hat. Now suppose the parents take away the whip and the knife and replace them with a bucket and a shovel. It may be a relief for them to know that they will either disappoint both children or delight them both. In other words, the parents may want to make sure that their action does not destabilize their childrens' bonding by making one child better off and the other worse off and thus giving the impression that parents favor one child over the other. On a larger scale, a government may take a similar stance when it comes to family policies: it may be desirable to know that a change in family policies such as changes in parental leave policies or family taxation, – policies that clearly change a family's production possibility set – do not leave some family members worse off and others better off, which could unduly stress intrafamily relationships.

In practice, however, most bargaining situations which draw upon the results of axiomatic bargaining typically do so using some generalized utilitarian bargaining solution (GUBS). This broad class of solutions consists of maximizing an additively separable social welfare function and includes the utilitarian and Nash bargaining solutions. Despite their appealing properties (Moulin 1988), bargaining solutions in the GUBS class typically fail to satisfy solidarity (Chun and Thomson, 1988) unless agents' utility is transferable. Yet, transferable utility (TU) is a very strong assumption which forbids taking into account commonly observed features of individual preferences such as diminishing marginal utility and, by extension, risk aversion. Hence, for most practical purposes, the use of the class of GUBS seems rather limited in applications.

We remedy the issue by establishing that well-behaved GUBS (to be defined) satisfy solidarity on a broader utility class than TU, which we define and call Almost TU. Almost TU requires the same ordinal properties as TU but allows for cardinal properties like diminishing marginal utility as well. Hence, our result broadens the valid range of applications for bargaining solutions of the GUBS class.[1] Also, from an implementation standpoint, policy makers may be unsure of the cardinal properties of agents' utility functions when evaluating the change in welfare due to a change in policy. Hence they may be reassured to know that the utility of the agents will change in the same direction even in the event of a misestimation of these cardinal properties, as long as the ordinal

[1] MasCollel et al. (1995, p. 831) take as given that individuals have cardinal utility functions, when they state:"[...] whereas a policy maker may be able to identify individual cardinal utility functions (from revealed risk behavior, say), it may actually do so but only up to a choice of origins and units."

properties for TU are satisfied.

More precisely, our main result (Theorem 1) characterizes Almost TU as the domain on which all well-behaved GUBS satisfy solidarity. We then present two applications where our main result bears useful consequences. First, in family policies, changes in family taxation amount to changes in the production possibility set of the household. Consquently, knowing that the solidarity property holds helps decrease the number of dimensions of possible opposition to a policy change (i.e., only interhousehold tensions will have to be considered, but not intrahousehold tensions). The second application we offer relates to the question of incentive compatibility (Theorem 2) and as an application we present a version of the Rotten Kid Theorem which holds on the wider domain of Almost TU instead of TU.

Even if a GUBS satisfies the solidarity property, there is still a possibility that a change in the utility possibility set affects agents' utilities in opposite ways. This happens if not only joint production possibilities change but the stand-alone utilities of agents change as well (see section 6 for examples). Yet, even in this case, Almost TU in combination with a GUBS remains useful as it allows us to decompose the impact of such a change into a "utility possibility set" effect (agents share the gain or the pain holding the disagreement point fixed) and a "disagreement point" effect (different agents may experience changes in their utility at the disagreement point in opposing directions).

2 Related Literature

Many works emphasize the importance of solidarity in allocation problems, be it with respect to population or to the total amount of goods available (see Moulin, 1988, or Sprumont, 2008, for a survey.) This work belongs to the latter strand of the literature and is more closely related to Chun and Thomson (1988), which explicits the parallel between fair allocation problems and bargaining situations. Chun and Thomson (1988) show that the solidarity property holds in a one-good economy for some bargaining solutions. Our main result generalizes theirs to a many-goods production economy when preferences exhibit Almost TU.

In the context of axiomatic bargaining, many characterizations of bargaining solutions are motivated by at least some notion of solidarity or monotonicity arguments.[2] Xu and Yoshihara (2008) offer a systematic treatment of well-known bargaining rules with respect to solidarity-type axioms.

Although we motivate the interest in the solidarity property as a normative issue, our result also has implications for incentive compatibility. In an application of our main theorem we draw a connection to Bergstrom's (1989) treatment of the Rotten Kid Theorem.[3] Bergstrom shows that each child behaves so as to

[2] Individual monotonicity is the property that distinguishes the Kalai-Smorodinsky solution from the Nash Bargaining solution (Kalai and Smorodinsky 1975).

[3] There has been renewed interest in the Rotten Kid Theorem in explaining the economics of child labor (Baland and Robinson 2000, and Bommier and Dubois 2004). The question of whether or not TU is a reasonable assumption in this context plays a crucial role in these two

maximize the head of household's altruistic utility function if and only if utility is transferable**.** By strengthening the assumption on the altruistic parent's preferences to be of the form of a well-behaved GUBS rather than just treating each child as a normal good, we present a Rotten Kid Theorem that holds for Almost TU, not just TU.

The issue of incentive compatibility also arises in a model in which spouses need to take individual actions in order to produce goods that they will later distribute among themselves. Based on Gugl (2009) we provide such a model of "Rotten Spouses" in section 6.1. Almost TU takes care of two issues at once: Assuming Almost TU, the class of well-behaved GUBS satisfies the Solidarity property as well as incentive compatibility.

3 The Model

Consider a population $N = \{1, 2, ..., n\}$ of agents who produce $L \geq 2$ goods. These goods may include public goods, but at least one good is private. More precisely, the population faces a *production possibility set* $Y \subset \mathbb{R}^L_+$ that is a closed, convex and comprehensive set. If we denote by $y \in \mathbb{R}^L_+$ a particular *product mix*, then ∂Y, the production possibility frontier of Y, and the corresponding *transformation function*, $F : \mathbb{R}^L_+ \to \mathbb{R}$, are defined as follows:

$$\begin{aligned} Y &= \left\{y \in \mathbb{R}^L_+ | F(y) \leq 0\right\}, \text{ and} \\ \partial Y &= \{y \in Y | F(y) = 0\}. \end{aligned}$$

We denote by $x_i = (x_{i1}, x_{i2}, ..., x_{iL}) \in \mathbb{R}^L_+$ agent i's *consumption vector*. A *distribution of* y is a list of consumption vectors, one per agent, $x = (x_1, ..., x_n)$ such that:

$$\begin{cases} \sum_{i \in N} x_{il} = y_l & \text{for any private good, } l, \text{ and} \\ x_{ik} = x_{jk} = y_k & \text{for all } i, j \in N \text{ and any public good, } k. \end{cases}$$

For any product mix y, we denote by $X(y)$ the *set of distributions of* y and by $X(Y) = \bigcup_{y \in Y} X(y)$ the set of *feasible* distributions under Y. An a*llocation* is a product mix-distribution pair $(y, x) \in Y \times X(y)$.

The preferences of each agent i are represented by a *utility function*, u_i, which is non-decreasing, concave and twice differentiable from $\mathbb{R}^L_+$ to $\mathbb{R}$.[4] We denote by $\mathcal{U}$ the class of such utility functions. A *utility profile* is a collection of utility functions, $(u_1, u_2, ..., u_n) \in \mathcal{U}^N$, one per agent. An *economy* is a pair $(Y, u) \in \mathbb{R}^L_+ \times \mathcal{U}^N$.

We denote by $U(Y, u) = \{\psi \in \mathbb{R}^N | \exists x \in X(Y) \text{ s.t. } u(x) = \psi\}$ the *utility possibility set* corresponding to the economy (Y, u). It follows from our assumptions on an economy that $U(Y)$ is a closed, convex, and comprehensive set.

papers.

[4]We assume differentiability for expositional purposes, but our results extend to Leontief-type preferences.

We denote by $\partial U(Y,u)$ the Pareto frontier of $U(Y,u)$; i.e., $\partial U(Y,u) = \{\psi \in U(Y,u) | \psi' \geq \psi \implies \psi' \notin U(Y,u)\}$.[5]

We shall consider that agents cooperatively manage the economy, in the form of a bargaining process, with the possibility that agents disagree on how to do so. Hence, we denote by d_i agent i's stand-alone utility level and call $d = (d_1, d_2, ..., d_n) \in U(Y,u)$ the *disagreement point* of the bargaining process. We denote by the pair $(U(Y,u), d)$ the corresponding *bargaining problem.* Note that we take the view that the disagreement point may depend on the utility profile, u, but is independent of the cooperative production possibilities, Y.

A bargaining solution is a function, $S : \mathbb{R}^L_+ \times \mathcal{U}^N \times \mathbb{R}^N \to U(Y,u)$, mapping to each bargaining problem a utility vector in the corresponding utility possibility set such that $S(U(Y,u),d) \geqq d$ and $S(U(Y,u),d) \in \partial U(Y,u)$. We denote by $\mathcal{S}$ the class of bargaining solutions. A family of bargaining solutions we shall consider is that of generalized utilitarian bargaining solutions (GUBS) where, for each bargaining solution Γ in this class, there exists a list of n concave, strictly increasing, and continuous functions, $(\gamma_1, \gamma_2, ... \gamma_n)$, such that $\Gamma(U(Y,u),d) = \arg\max_{\psi \in \partial U(Y,u)} \sum_{i \in N} \gamma_i(\psi_i - d_i)$. Note that GUBS is a wide family of bargaining solutions; both the utilitarian solution and the Nash bargaining solution belong to GUBS, with $\gamma_i = 1$ and $\gamma_i = \ln(\cdot)$ for all i, respectively. More precisely, we denote by $\mathcal{G}$ the subclass of GUBS for which the γ_i's are strictly concave; thus, so the Nash Bargaining solution belongs to $\mathcal{G}$. The utilitarian solution belongs to another subclass of GUBS, the weighted utilitarian bargaining solutions (WUBS). A bargaining solution W belonging to WUBS is characterized by a list of n non-negative weights, $(\omega_1, \omega_2, ..., \omega_n) \in \mathbb{R}^N_+$, with $\sum \omega_i = 1$, such that $W(U(Y,u),d) \in \arg\max_{\psi \in \partial U(Y,u)} \sum_{i \in N} \omega_i(\psi_i - d_i)$. To ensure uniqueness of the solution in case $\partial U(Y,u)$ is an $(n-1)$-dimensional hyperplane as Pareto frontier, we shall consider only the subfamily of WUBS which break ties along a non-decreasing path of $\mathbb{R}^N_+$. A *non-decreasing path* is a function $\pi : \mathbb{R}_+ \to \mathbb{R}^N_+$ s.t. $\pi(t) = (\pi_1(t), \pi_2(t), ..., \pi_n(t))$ non-decreasing in each coordinate with $\pi(0) = d$ and $\sum_i \pi_i(t) = t + \sum_i d_i$.

$$W(U(Y,u),d) = \arg \min_{\psi \in \arg\max_{\psi \in \partial U(Y,u)} \sum \omega_i(\psi_i - d_i)} \left\| \psi - \pi\left(\max_{\psi \in \partial U(Y,u)} \sum \omega_i(\psi_i - d_i) \right) \right\|$$

By convexity of $U(Y,u)$, $W(U(Y,u),d)$ is unique. We denote by $\mathcal{W}$ the family of WUBS breaking ties along a non-decreasing path. We refer to a $\mathcal{G} \cup \mathcal{W}$ as the class of *well-behaved GUBS*.

We say that a bargaining solution, S, satisfies *Solidarity under utility profile* u if one of the following vector inequality holds:

$$S(U(Y,u),d) \geqq S(U(Y',u),d) \quad \text{or} \quad S(U(Y,u),d) \leqq S(U(Y',u),d)$$

for any $Y, Y' \subset \mathbb{R}^L_+$, and any $d \in \mathbb{R}^N_+$.

[5] We adopt the ususal notational convention for vector inequalities: $x \geqq x'$, $x \geq x'$, and $x > x'$.

4 Almost Transferable Utility

In this section we define the new concept of Almost Transferable Utility. In order to do so, we first recall what is meant by a product mix that being efficient independently of distribution, a necessary (and sufficient) condition for (Almost) TU to hold. The conditions under which a product mix is efficent independently of distribution are well established in the literature by Bergstrom and Cornes (1983) and Bergstrom and Varian (1985). Our analysis below is not meant to reestablish these results but to apply the concept to introduce Almost TU.

4.1 Efficiency Independent of Distribution[6]

We denote by $EX(y,u) = \{x \in X(y) | \nexists x' \in X(y), u(x') \geq u(x)\}$ the set of exchange efficient distributions in $X(y)$ relative to the utility profile u and by $P(Y,u) = \{(y,x) \in Y \times X(y) | \nexists (y',x') \in Y \times X(y'), u(x') \geq u(x)\}$ the set of (Pareto) efficient allocations in the economy (Y,u).

For any given $y \in Y$, we say that $x \in X(y)$ is *interior exchange efficient* if and only if:

$$\frac{\frac{\partial u_i(x_i)}{\partial x_{li}}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(x_j)}{\partial x_{lj}}}{\frac{\partial u_j(x_j)}{\partial x_{mj}}} \quad \text{for all } i,j \in N, \text{ for all private goods } l,m.$$

We denote by $EX^*(y,u)$ the set of interior exchange efficient distributions of $X(y)$ relative to the utility profile u. We say that a product mix and distribution pair $(y,x) \in Y \times X(y)$ is *interior efficient* if and only if the following holds:

$$\begin{cases} \qquad \frac{\frac{\partial u_i(x_i)}{\partial x_{li}}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} = \frac{\frac{\partial F(y)}{\partial y_l}}{\frac{\partial F(y)}{\partial y_m}} & \text{for all } i \in N, \text{ and all private goods } l,m \\ \text{and} \quad \sum_{i \in N} \left(\frac{\frac{\partial u_i(x_i)}{\partial x_k}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} \right) = \frac{\frac{\partial F(y)}{\partial y_k}}{\frac{\partial F(y)}{\partial y_m}} & \text{for any public good } k \text{ and any private good } m. \end{cases} \tag{1}$$

The first family of equalities states that all agents' marginal rates of substitutions (MRS) between any two private goods, $\frac{\partial u_i(x_i)}{\partial x_{li}} / \frac{\partial u_i(x_i)}{\partial x_{mi}}$, must equal the marginal rate of transformation (MRT) between these goods, $\frac{\partial F(y)}{\partial x_l} / \frac{\partial F(y)}{\partial x_m}$. The second set of equalities are the Samuelson conditions for public goods.

We say that *efficiency is independent of distribution* if there exists $\overline{y} \in Y$ such that, for all $\psi \in \partial U(Y,u)$, $\psi = u(x)$ for some $x \in X(\overline{y})$. In words, all points on the utility possibility frontier are be achieved via vasiour distributions of the same product mix. It follows from (1) that $\bar{y}$ is associated with a vector of marginal rates of substitution at the exchange efficient distributions—one per pair of goods—which is independent of the utility level achieved by any agent. See Figure 1 for a graphical illustration in the case of two private goods, strictly quasi-concave utility functions and a strictly convex production possibility set.

[6] Bergstrom and Cornes (1983) call this concept "independence of allocative efficiency from distribution."

In the figure, neither y nor y' are efficient, because the value of the MRS associated with $EX^*(y,u)$ is different from the MRT at y; similarly for y'. The product mix associated with any efficient allocation, y'', lies on ∂Y between y and y', where the MRS of $EX^*(y'',u)$ equals the MRT at y''.

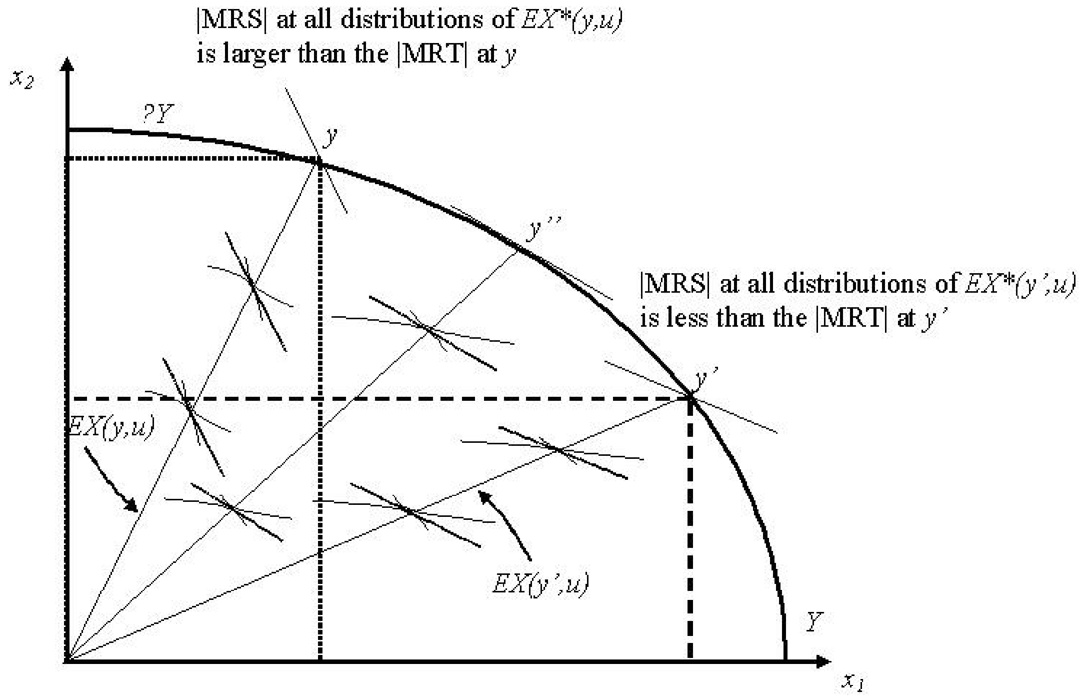

Figure 1: Effficicency independent of distribution: Anll efficient distributions aggregate up to the same product mix y''.

In the case of public and private goods, for a product mix to be efficient independently of distribution requires that the sum of marginal rates of substitution stays the same for a given level of the public good $\bar{y}_k$ no matter how the total amount of the private good $\bar{y}_l$ is distributed.

4.2 Transferable Utility

A utility profile, $u \in \mathcal{U}^N$, satisfies *Transferable Utility* (Bergstrom 1989) if for any given $Y \in \mathbb{R}_+^L$, the following holds:

$$\partial U(Y,u) = \{\psi \in U(Y,u) : \sum_{i \in N} \psi_i = \lambda(Y,u)\}.$$

Note that resource and technological constraints as given by Y only play a role in the size of λ: If TU holds, efficiency is independent of distribution (Bergstrom and Cornes 1983, and Bergstrom and Varian 1985). If we take agents' utility to be ordinal, the converse is also true. Bergstrom and his co-authors give an exhaustive list of agents' utility functions that lead to TU. Agents' utility functions must allow the indirect utility representation of the Gorman Polar Form in an economy with only private goods (Bergstrom and Varian 1985) and a form dual to the Gorman Polar form in an economy with public and private goods (Bergstrom and Cornes 1981 and 1983).

Example 1 *Finding the utility possibility frontier with TU.*

a) Two private goods, two agents. Suppose $u_i = (x_{1i}x_{2i})^{1/2}$ *. Then* $\partial U(Y,u) = \{(\psi_1,\psi_2) \in U(Y,u) : \psi_1 + \psi_2 = \lambda(Y,u)\}$*, where* $\lambda = \max_{y\in Y}(y_1y_2)^{1/2}$*.Indeed, because both agents have identical preferences for any given* $y \in Y$*, dividing all goods equally must be exchange efficient, i.e.* $x = \left(\frac{1}{2}y_1, \frac{1}{2}y_2, \frac{1}{2}y_1, \frac{1}{2}y_2\right) \in EX^*(y,u)$*. Therefore* $\lambda(y,u) = \frac{1}{2}(y_1y_2)^{1/2} + \frac{1}{2}(y_1y_2)^{1/2} = (y_1y_2)^{1/2}$ *and* $\lambda(Y,u) = \max_{y\in\partial Y}(y_1y_2)^{1/2}$*.*

b) A private and a public good, two agents. Suppose preferences over a private good and a public good are quasi-linear such that $u_i = x_{1i} + h_i(x_2)$*, where* $h_i(\cdot)$ *is a strictly concave function. Then the segment of the utility possibility frontier at which TU holds consists of all the points on the line from* $(\psi_1 = h_1(y_2), \psi_2 = y_1 + h_2(y_2))$ *to* $(\psi_1 = y_1 + h_1(y_2), \psi_2 = h_2(y_2))$ *where the vector* (y_1,y_2) *is found by* $\arg\max_{y\in Y} y_1 + h_1(y_2) + h_2(y_2)$*, and* $\lambda = \max_{y\in Y} y_1 + h_1(y_2) + h_2(y_2)$*.*

More generally, when TU holds, one can find $\partial U(Y,u)$ in two steps. First, calculate $\lambda(y,u) = \sum_{i\in N} u_{i(x_i)}$ such that $x \in EX^*(y,u)$. Second, find $\lambda(Y,u) = \max_{y\in\partial Y}\lambda(y,u)$.

4.3 Almost Transferable Utility

Whether a product mix is efficient independently of distribution, depends solely on the ordinal properties of the agents' utility functions. If they are such that a product mix is efficient independently of distribution, but their cardinal properties prohibit the particular utility representation that would lead to TU, then there must exist positive monotonic transformations,$f_i : \mathbb{R} \to \mathbb{R}$, such that

$$\sum_{i\in N} f_i(\psi_i) = \lambda(Y, f(u)).$$

However, $f_i(\psi_i)$ no longer represents an agent's utility. Hence, we say that profile $u \in U^N$ exhibits *Almost Transferable Utility (Almost TU)* if, for any given Y, the utility possibility frontier is of the form

$$\partial U(Y,u) = \{\psi \in U(Y,u) : \sum_{i\in N} f_i(\psi_i) = \lambda(Y, f(u))\}.$$

Since u_i is assumed to be concave, it follows that f_i must be an increasing, and convex function. The intuition is that in order to recover a linear constraint, one needs constant "marginal utility" of money. Hence, given that strict concavity of u_i leads to decreasing marginal utility of money, a strictly convex transformation is required to undo this effect.

Example 2 *Finding the utility possibility frontier with Almost TU.*

a) Take the same ordinal properties as in Example 1, but different cardinal properties. Suppose $u_1 = (x_{11}x_{21})^{1/3}$ *and* $u_2 = (x_{12}x_{22})^{1/4}$, *that is, given* u *from example 1a),* $u = \left(u_1^{2/3}, u_2^{1/2}\right)$. *Then* $\partial U(Y,u) = \{(\psi_1, \psi_2) \in \mathbb{R}_+^2 : \psi_1^{3/2} + \psi_2^2 = \lambda\left(Y, u_1^{3/2}, u_2^2\right)\}$, *where* $\lambda = \max_{y \in Y}(y_1 y_2)^{1/2}$.

b) Suppose the cardinal utility function of agent i *over a private good* (x_{1i}) *and a public good* (x_2) *is given by* $u_i = (x_{1i} + h_i(x_2))^{\alpha_i}$, *where* $h_i(\cdot)$ *is a strictly concave function and* $\alpha_i \in (0,1)$. *Then the segment of the utility possibility frontier at which Almost TU holds consists of the endpoints* $\left(\psi_1 = (h_1(y_2))^{a_1}, \psi_2 = (y_1 + h_2(y_2))^{\alpha_2}\right)$ *and* $\left(\psi_1 = (y_1 + h_1(y_2))^{\alpha_1}, \psi_2 = (h_2(y_2))^{\alpha_2}\right)$ *and of all points* (ψ_1, ψ_2) *between these endpoints for which* $\psi_1^{1/\alpha_1} + \psi_2^{1/\alpha_2} = \lambda\left(Y, u_1^{1/\alpha_1}, u_2^{1/\alpha_2}\right)$, *where the vector* (y_1, y_2) *is found by* $\arg\max_{y \in Y} y_1 + h_1(y_2) + h_1(y_2)$, *and* $\lambda = \max_{y \in Y} y_1 + h_1(y_2) + h_1(y_2)$.

In the following example (Almost) TU does not hold; although there exists a Y such that $U(Y,u)$ forms a simplex, it is not the case for all production possibility sets.

Example 3 *Three private goods, two agents. Let* u_1 *and* u_2 *be strictly increasing, strictly quasi-concave and homogenous of degree one. Moreover, let* $u_1 = u(x_{11}, x_{21})$, $u_2 = u(x_{12}, x_{32})$. *That is, good 3 replaces good 2 in agent's 2 utility function as compared to that of agent 1's. Let* $\lambda(Y, u_1) = \max_{y \in Y} u(x_{11}, x_{21})$ *and* $\lambda(Y, u_2) = \max_{y \in Y} u(x_{12}, x_{32})$. *Consider the production possibility set given by* $Y = \left\{(y_1, y_2, y_3) \in \mathbb{R}_+^3 | y_1 + p_2 y_2 + p_3 y_3 \leq I\right\}$ *with* $p_2, p_3, I > 0$ *and* $p_2 = p_3 = p$. *Then,* $\lambda(Y, u_1) = \lambda(Y, u_2)$ *and* $\partial U(Y,u) = \{(\psi_1, \psi_2) \in U(Y,u) : \psi_1 + \psi_2 = \lambda(Y, u_1) = \lambda(Y, u_2)\}$. *Suppose* Y *changes to* Y' *such that* $Y' = \left\{(y_1, y_2, y_3) \in \mathbb{R}_+^3 | y_1 + p_2 y_2 + p_3 y_3 \leq I\right\}$ *with* $p_2 \neq p_3$. *Then* $\partial U(Y', u) = \{(\psi_1, \psi_2) \in U(Y', u) : \frac{\psi_1}{\lambda(Y', u_1)} + \frac{\psi_2}{\lambda(Y', u_2)} = 1\}$. *(Almost) TU is violated. Proof see Appendix.*

5 Results

We now consider what happens if Almost TU holds and the production possibility set changes.

Lemma Given Almost TU, a change in the production possibilities of the economy can only result in an expansion or a contraction of the utility possibility set:$\forall Y, Y' \in \mathbb{R}_+^L$, $[U(Y,u) \subseteq U(Y',u)$ or $U(Y',u) \subseteq U(Y,u)]$.

Proof. Consider a change from Y to Y'. By Almost TU, $\partial U(Y,u) = \{\psi \in U(Y,u) : \sum f_i(\psi_i) = \lambda(Y, f(u))\}$ and $\partial U(Y',u) = \{\psi \in U(Y',u) : \sum f_i(\psi_i) = \lambda(Y', f(u))\}$. Hence, either $\lambda(Y, f(u)) = \lambda(Y', f(u))$, in which case $U(Y,u) = U(Y',u)$; or $\lambda(Y, f(u)) > \lambda(Y', f(u))$ implying $U(Y,u) \supset U(Y',u)$; or $\lambda(Y, f(u)) < \lambda(Y', f(u))$ implying $U(Y,u) \subset U(Y',u)$. ■

We now state our main theorem.

Theorem 1 Any bargaining solution, $S \in \mathcal{G} \cup \mathcal{W}$, satisfies solidarity under profile u if and only if u exhibits Almost TU.

Proof. For sufficiency, note that for any $S \in \mathcal{G} \cup \mathcal{W}$, a first step—and, in case of $S \in \mathcal{G}$, also the last step—to finding S is by solving the following:

$$\begin{array}{rc} & \max_\psi \sum_{i\in N} \gamma_i(\psi_i - d_i) \\ s.t. & \sum_{i\in N} f_i(\psi_i) = \lambda(Y, f(u)) \end{array} \tag{2}$$

In what follows, it will be useful to work with $v_i = f_i(\psi_i)$, such that the above problem becomes

$$\begin{array}{rc} & \max_\psi \sum_{i\in N} \gamma_i(f_i^{-1}(v_i) - d_i) \\ s.t. & \sum_{i\in N} v_i = \lambda(Y, f(u)) \end{array} \tag{3}$$

and then finding the corresponding ψ_i from $\psi_i = f_i^{-1}(v_i)$. Chun and Thomson (1988) show that if agents have concave utility functions over one good only, and this good's supply increases, both agents benefit under the Nash bargaining solution. The authors remark that the result extends to any bargaining solution in $\mathcal{G}$ (Chun and Thomson 1988, p. 19). When problem (2) is presented as (3), a change in $\lambda(Y, f(u))$ due to a change in Y has the same impact on v as a change in the only good has on agents' utilities in Chun and Thomson (1988)'s one-good economy. Thus their proof applies to our problem (3) for any $G \in \mathcal{G}$ such that $\lim_{x_i \to 0} g_i = -\infty$**.** In addition, our proof of sufficiency below also handles the subclass $\mathcal{W}$ of GUBS, the possibility of corner solutions[7]**,** and accounts for $d \neq (0,0)$.

Suppose Almost TU holds, and let $S \in \mathcal{G} \cup \mathcal{W}$, $Y \subset \mathbb{R}^L_+$, $u \in \mathcal{U}^N$ and $d \in U(Y,u)$. Denote by $v^* = S(\{v \in \mathbb{R}^N | \sum_{i\in N} v_i \leq \lambda(Y, f(u))\}, d)$ the solution vector in the v-space.

- Case 1: $\frac{df_i^2}{d\psi_i^2}(f_i^{-1}(v_i^*)) > 0$ for at least $n-1$ values of $i \in N$. It follows that v^* is the unique element of $\partial f(U(Y,u))$ such that the following expression holds:

$$\left\{ \begin{array}{l} \frac{\partial \gamma_i}{\partial\left(f_i^{-1}(v_i) - d_i\right)} \frac{df_i^{-1}}{dv_i}(v_i) = \frac{\partial \gamma_j}{\partial\left(f_j^{-1}(v_j) - d_j\right)} \frac{df_j^{-1}}{dv_j}(v_j) \qquad \text{for all } i,j \in N \\ \text{such that } f_i^{-1}(v_i) > \max\left(\underline{\psi_i}, d_i\right) \text{ and } f_j^{-1}(v_j) > \max\left(\underline{\psi_j}, d_j\right), \end{array} \right. \tag{4}$$

[7]The possibility of corner solutions is not a concern in the proof of Chun and Thomson (1988) as the Nash bargaining solution is necessary interior.

where $\underline{\psi_i}$ (resp. $\underline{\psi_j}$) is the utility level of agent i (resp. agent j) when she does not receive any private good. The left hand side of (4) depends on v_i only, and the right hand side of (4) depends on v_j only. Since $f_i^{-1}(\cdot)$ is increasing and concave, $\frac{df_i^{-1}}{dv_i}$ is positive and non-increasing. Similarly, γ_i being concave and f_i^{-1} being increasing, $\frac{\partial\gamma_i}{\partial\left(f_i^{-1}(v_i)-d_i\right)}$ is non-increasing in v_i. Hence, the product $\frac{\partial\gamma_i}{\partial\left(f_i^{-1}(v_i)-d_i\right)}\frac{df_i^{-1}}{dv_i}$ (resp.$\frac{\partial\gamma_j}{\partial\left(f_j^{-1}(v_j)-d_j\right)}\frac{df_j^{-1}}{dv_j}$) is non-increasing in v_i (resp.v_j). Therefore, for (4) to hold as λ changes values, v_i and v_j must change in the same direction, thus proving the result.

- Case 2: $\frac{d^2 f_i}{dv_i^2}(f_i^{-1}(v_i^*)) > 0$ for at most $n-2$ values of $i \in N$. If $S \in \mathcal{G}$, v^* is the unique element of $\partial f(U(Y,u))$ for which expression (4) holds and the argument of Case 1 follows through. Now, suppose $S \in \mathcal{W}$, with $\omega \in \mathbb{R}^n_+$ its associated weights, there may be more than one $v \in \partial f(U(Y,u))$ for which expression (4) holds. Denote

$$\sigma(Y,u,\omega) = \left\{ \begin{array}{c} \psi \in \partial U(Y,u)| \\ \omega_i \times \frac{df_i^{-1}}{\partial v_i}(v_i) = \omega_j \times \frac{df_j^{-1}}{dv_j}(v_j) \qquad \text{for all } i,j \in N \\ \text{and } \psi_i > \max\left(\underline{\psi_i}, d_i\right) \text{ and } \psi_j > \max\left(\underline{\psi_j}, d_j\right). \end{array} \right\}.$$

  We proceed to show that the fact that $\sigma(Y,u,\omega)$ may not be a singleton does not affect the comonotonicity of the utility shares. In particular, despite the fact that S breaks ties along a non-decreasing path, this is not automatic as the path may not pass through $\sigma(Y,u,\omega)$.

  It follows from elementary convex optimization arguments that if $\sigma(Y,u,\omega)$ is not a singleton, then $\lim_{\hat{\omega}_t\to\omega}\sigma(Y,u,\hat{\omega}_t)$ is a singleton for any sequence of $\mathbb{R}^n_+$, $\{\hat{\omega}_t\}_{t\in\mathbb{N}}$, such that $\hat{\omega}_t \neq \omega$ for all t and $\lim_{t\to\infty}\hat{\omega}_t = \omega$. Therefore, an argument similar to that in Case 1, applied to the sequences $\{\hat{\omega}_t\}_{t\in\mathbb{N}}$ implies that, for any $Y' \subset \mathbb{R}^L_+$ such that, by Lemma 5, $U(Y,u) \subset U(Y',u)$ (resp $U(Y',u) \subset U(Y,u)$), the set $\sigma(Y',u,\omega)$ dominates (resp. is dominated by) $\sigma(Y,u,\omega)$ in the following sense: for any element $\psi \in \sigma(Y,u,\omega)$, there exists $\psi' \in \sigma(Y',u,\omega)$ such that $\psi' \geq \psi$ (resp. $\psi' \leq \psi$). See Figure 2. Thus, the fact that S breaks ties along a non-decreasing path yields the desired result, regardless of whether this path passes through $\sigma(Y,u,\omega)$ or $\sigma(Y',u,\omega)$.

For necessity, let S be a generalized utilitarian bargaining solution and let $u \in \mathcal{U}^n$ be a utility profile which does not satisfy Almost TU. Consider a production possibility set, $Y_1 \subset \mathbb{R}^L_+$, and let $y \in \partial Y_1$ be an efficient product mix, so that $a \in EX^*(y,u)$ in the economy (Y_1,u). By efficiency:

$$\frac{\frac{\partial u_i(a_i)}{\partial x_{li}}}{\frac{\partial u_i(a_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(a_j)}{\partial x_{lj}}}{\frac{\partial u_j(a_j)}{\partial x_{mj}}} = \frac{\frac{\partial F_1(y)}{\partial y_l}}{\frac{\partial F_1(y)}{\partial y_m}} \tag{5}$$

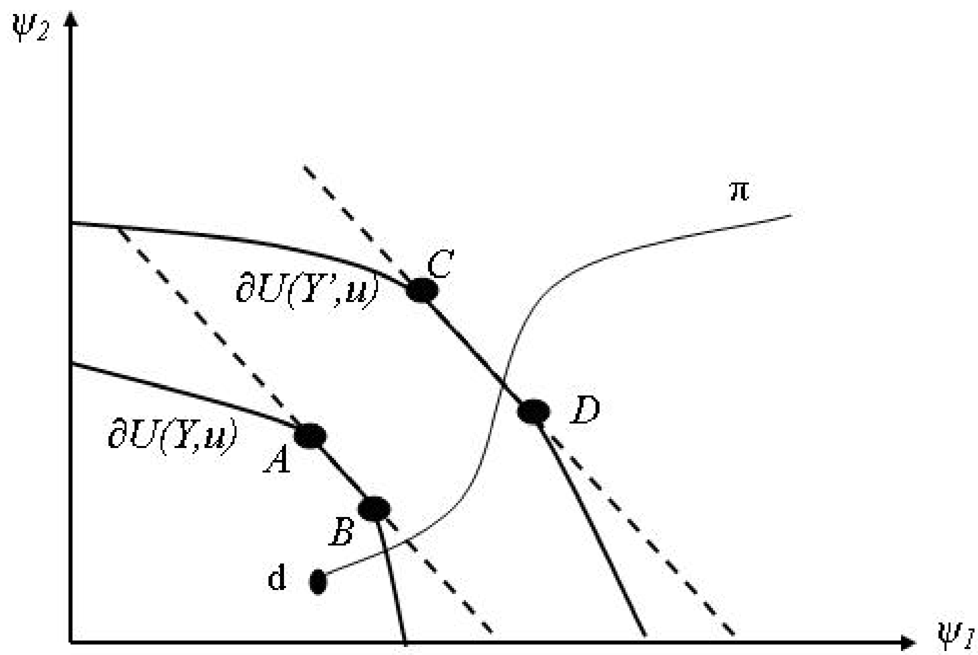

Figure 1: By Case 1, $A < C$ and $B < D$.

for any pair of agents i and j and any pair of goods l and m. [8]

By continuity, and because the profile u does not satisfy Almost TU, there exists an interior exchange efficient distribution $b \in EX^*(y,u)$ such that (y,b) is not Pareto efficient in the economy (Y_1,u). Therefore, $\frac{\frac{\partial u_i(b_i)}{\partial x_{li}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{lj}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}}$ for all $i,j \in N$ and all $l,m \in L$ but $\frac{\frac{\partial u_i(b_i)}{\partial x_{li}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{lj}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}} \neq \frac{\frac{\partial F_1(y)}{\partial y_l}}{\frac{\partial F_1(y)}{\partial y_m}}$ for some pair l,m of goods. Without loss of generality, suppose that

$$\frac{\frac{\partial u_i(b_i)}{\partial x_{1i}}}{\frac{\partial u_i(b_i)}{\partial x_{2i}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{1j}}}{\frac{\partial u_j(b_j)}{\partial x_{2j}}} > \frac{\frac{\partial F_1(y)}{\partial y_1}}{\frac{\partial F_1(y)}{\partial y_2}},$$

for all $i,j \in N$.

Now construct another production possibility set, $Y_2 \subset \mathbb{R}^L_+$, such that $y \in \partial Y_2$ and $\frac{\frac{\partial F_2(y)}{\partial y_l}}{\frac{\partial F_2(y)}{\partial y_m}} = \frac{\frac{\partial u_i(b_i)}{\partial x_{li}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{lj}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}}$ for all $l,m \in L$ and all $i,j \in N$, as shown in Figure 3 in the two-agent case.

[8] For clarity, we are presenting the proof in the case of all private goods. The proof with public goods is similar, with the efficiency condition (5) being replaced by the Samuleson condition for these goods (Expression (1)).

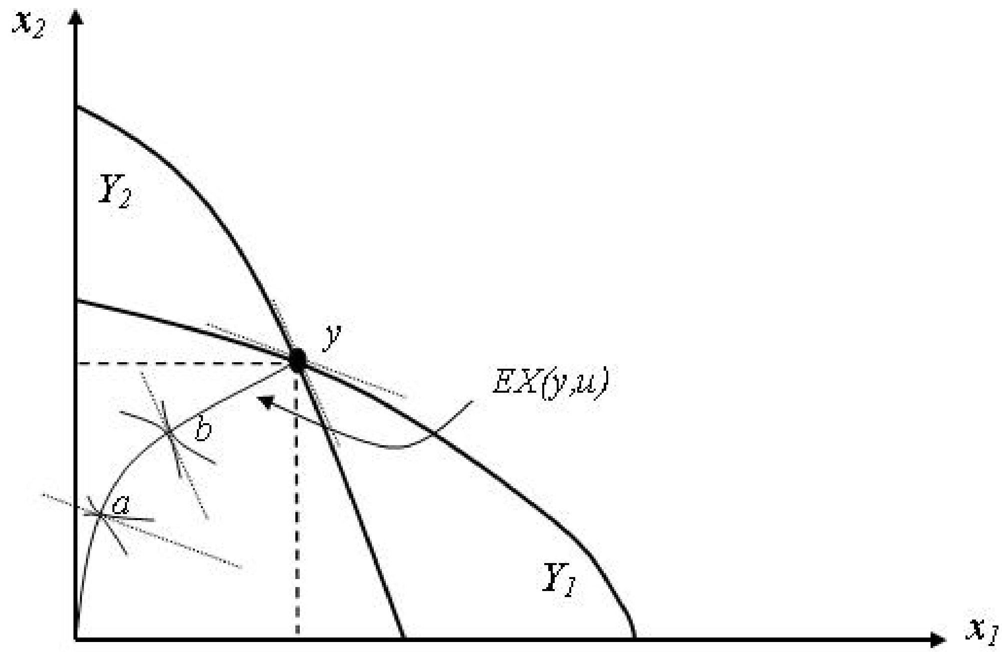

Figure 3: Allocation a (b) is efficient when the production set is Y_1 (Y_2).

It follows from the construction of Y_2 that a is not an efficient allocation in (Y_2, u) because $\frac{\frac{\partial F_2(y)}{\partial y_1}}{\frac{\partial F_2(y)}{\partial y_2}} > \frac{\frac{\partial u_i(a_i)}{\partial x_{1i}}}{\frac{\partial u_i(a_i)}{\partial x_{2i}}}$ for all $i \in N$. Therefore, there exists an allocation in Y_2 which Pareto-dominates a. In other words, if we denote by ψ_{a1} the utility vector corresponding to distribution a, (recall that efficiency of a in the economy (Y_1, u) implies that $\psi_{a1} \in \partial U(Y_1, u)$) there exists another vector $\psi_{a2} \in \partial U(Y_2, u)$ such that $\psi_{a2} > \psi_{a1}$. See Figure 4.

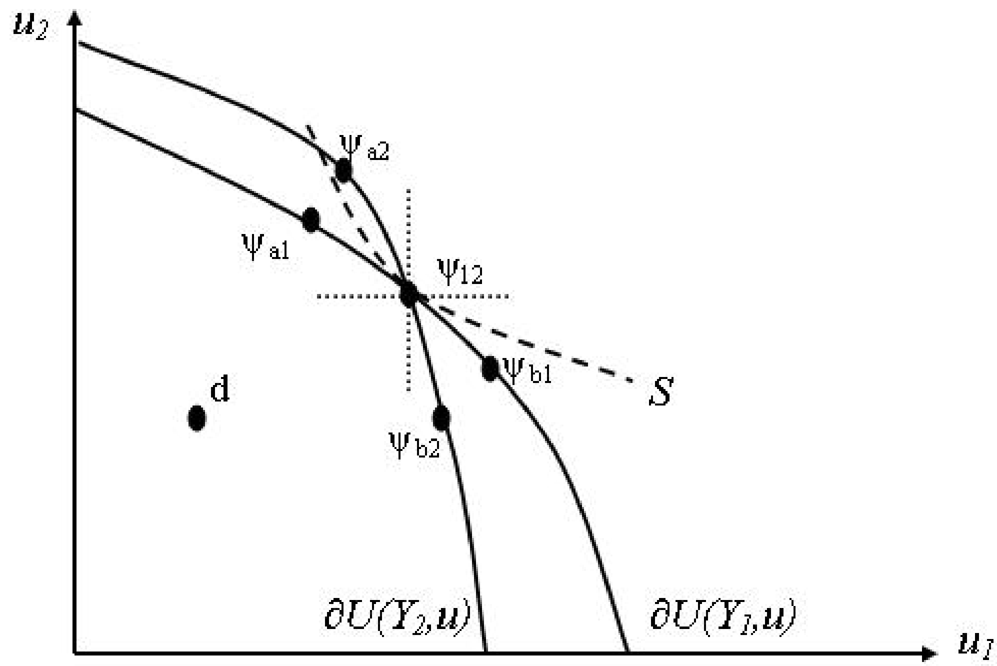

Figure 4: When utility possibility frontiers cross, solidarity may not hold

Similarly, because the allocation $(y, b) \in P(Y_2, u) \backslash P(Y_1, u)$, there exists a utility vector $\psi_{b1} \in \partial U(Y_1, u)$ which dominates $\psi_{b2} \in \partial U(Y_2, u)$; i.e. $\psi_{b1} > \psi_{b2}$.

From the two previous arguments, and from the continuity of $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$, it must be that $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$ cross at some point in the utility space. Denote by $\psi_{12} \in \partial U(Y_1, u) \cap \partial U(Y_2, u)$ such a point.

We now show that there exist bargaining situations where a change from the production possibility set Y_1 to Y_2 will benefit some agents while hurting others. Consider a disagreement point, $d \in U(Y_1, u)$, such that $S(U(Y_1, u), d) = \psi_{12}$; i.e., such that $\psi_{12} = \arg\max_{\psi \in U(Y_1, u)} \sum_i \gamma_i(\psi_i - d_i)$. Such a disagreement point exists due to the continuity and the strict monotonicity and concavity properties of the γ_i's, if $S \in \mathcal{G}$ or, if $S \in \mathcal{W}$, due to the fact that S. breaks ties along a path. Therefore, invoking again the strict monotonicity and concavity of the γ_i's, and the fact that $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$ cross at ψ_{12}, it follows that $S(U(Y_2, u), d) \neq \psi_{12}$.[9] Finally, it follows from the fact that $U(Y_2, u)$ is convex and comprehensive that $S(U(Y_2, u), d)$ neither dominates nor is dominated by $S(U(Y_1, u), d)$. ■

[9]Recall that we defined the disagreement point to be entirely determined by the utility profile and the agents' stand-alone utilities. According to this interpretation, the disagreement point is unaffected by changes in the joint production possibilities.

Remark 4 *Many GUBS do not belong to $\mathcal{G} \cup \mathcal{W}$, where the $\gamma_i' s$ are not strictly concave everywhere, for which the proof of Theorem 1 readily applies.*

Remark 5 *Note that Chun and Thomson (1988)'s one-good economy is a special case of our economy; with $L = 1$, and y_1 given, it follows that $\sum_{i \in N} v_i = \sum x_{1i}$ and $\lambda(y_1, f(u)) = y_1$.*

Remark 6 *In the case of identical utility functions and a symmetric GUBS, if d is on the 45 degree line, the solidarity property is satisfied even if the bargaining solution is not well-behaved. By symmetry of $U(\cdot, u)$ it is impossible that $U(Y_1, u)$ and $U(Y_2, u)$ cross at the 45 degree line, yet a symmetric GUBS will always select ψ where $\partial U(\cdot, u)$ crosses the 45 degree line.*

6 Applications

Theorem 1 has important policy implications. For example, research on family economics frequently uses bargaining rules – most often the Nash bargaining solution – to analyze intrafamily distribution. In this literature parameters that change the disagreement point without changing the utility possibility set (McElroy (1990) refers to them as extrahousehold environmental parameters) have received substantial attention (Lundberg et al., 1997, Rubalcava and Thomas, 2000, Chiappori et al., 2002), but policies that have the potential to affect the disagreement point as well as the utility possibility set are more difficult to analyze. Examples of policies affecting the utility possibility set and *maybe* the disagreement point are parental leave policies, policies subsidizing child care, and family taxation. We focus on the latter in the application below.

6.1 Change in Family Tax Policy

Many tax expenditures and provisions in income taxation have a quite complex impact on a family's full budget set. For example the question of joint or individual taxation changes the household's production function of income. To fix ideas consider the following model based on Gugl (2009).

Consider a household consisting of two spouses ($i = f, m$) as the set of agents. Each spouse cares about his or her consumption of a private good (x_{1i}) and consumption of a household public good (x_2). Hence, a spouse's utility function is given by $u_i(x_{1i}, x_2)$. Total time endowment of each spouse is denoted by T, which can be divided between employment (l_i) towards purchasing the private good ($x_{1f} + x_{1m} = y_1$) and home production (t_i) to produce y_2.[10] Then a spouse's time constraint is given by $l_i + t_i = T$.

Household production is given by

$$x_2 = y_2(l_f, l_m) = h_f(T - l_f) + h_m(T - l_m). \tag{6}$$

[10] Home production can be interpreted as raising children, but can also stand for taking care of household chores like cooking, doing laundry, cleaning the house, gardening etc.

where h_f and h_m both satisfy the following properties: $h_i(0) = 0$, $\frac{\partial h_i}{\partial t_i} > 0$, and $\frac{\partial^2 h_i}{\partial t_i^2} \leq 0$.

Household Net Income A person's taxable income is given by the product of his or her wage rate (w_i) times the hours worked (l_i). Let $\phi^{ind}(w_i l_i)$ be the net wage of a person if taxed individually. Given a progressive tax on wages, $\phi^{ind}(.)$ is a strictly increasing and concave function. Thus $\frac{\partial \phi^{ind}}{\partial w_i l_i} w_i$ is the marginal net-wage rate of spouse i.[11]

Under joint taxation the net wage of the family as a whole is given by $\phi^{joint}(w_f l_f + w_m l_m)$. Given a progressive tax on wages, $\phi^{joint}(.)$ is a strictly concave function of the household's wage, $\sum_{i=f,m} w_i l_i$. Then $\frac{\partial \phi^{joint}}{\partial\left(\sum_{i=f,m} w_i l_i\right)} w_i$ is the marginal net-wage rate of spouse i under joint taxation.[12]

Normalizing the price of the private good to 1, the household budget constraint is given by

$$x_{1f} + x_{1m} = y_1 = \phi^{ind}(w_f l_f) + \phi^{ind}(w_m l_m) \tag{7}$$

under individual taxation, and by

$$x_{1f} + x_{1m} = y_1 = \phi^{joint}(w_f l_f + w_m l_m) \tag{8}$$

under joint taxation.

The couple's production possibility frontier, ∂Y, is found by

$$\max(y_1, y_2)$$

subject to constraints (6) and (7) under individual taxation and (6) and (8) under joint taxation. A change from individual to joint taxation is a rather complex change and it is possible that ∂Y^{ind} and ∂Y^{joint} intersect.[13]

Let the disagreement point be determined by the stand-alone utility of each spouse, i.e. a person's utility before marriage. Thus the tax schedule applied in the disagreement point is individual taxation and does not change with a change in family taxation.[14] Pollak (2006) argues that even if the disagreement point is determined by a non-cooperative game of spouses and not by the stand-alone utility, it should not change with a change in family taxation. He also concludes that "joint taxation provides incentives for specialization but [...] the

[11] Since the tax schedule starts at a zero tax rate and then increases the tax rate with wage income, $0 < \frac{\partial \phi^{ind}}{\partial w_i l_i} \leq 1$ for any $w_i l_i > 0$ and $\frac{\partial \phi^{ind}(0)}{\partial w_i l_i} = 1$.

[12] Again, the tax schedule starts at a zero tax rate and then increases the tax rate with wage income, so that $0 < \frac{\partial \phi^{joint}}{\partial\left(\sum_{i=f,m} w_i l_i\right)} \leq 1$ for any $w_i l_i > 0$ and $\frac{\partial \phi^{joint}(0)}{\partial \sum_{i=f,m} w_i l_i} = 1$.

[13] For example, in the US different tax schedules exist for singles and couples and a couple's tax liability is typically more than twice the amount of the tax liability of a single person earning half as much as the family as a whole. This form of joint taxation can lead to Y^{joint} to be neither a subset of, nor contain, Y^{ind}.

[14] See e.g. Gugl (2009) for such a specification.

distributional effects of joint taxation, which operate through the feasible set, are indeterminant" (Pollak 2006, p. 29). Assuming ATU offers a less ambiguous answer.

Proposition 1 If spouses have utility functions that lead to Almost TU and use a well-behaved GUBS to determine intrafamily distribution, both people either benefit or lose jointly with a change from individual taxation to joint taxation.

The result presented here establishes conditions under which a change in family taxation is guaranteed to change each family member's welfare in the same direction provided the disagreement point remains the same. Even if the disagreement point changes, as this would be the case if the tax schedule for singles also changes as part of a fundamental tax reform, Almost TU allows us to decompose the impact of a change in income taxation into a "utility possibility set" effect (family members share the gain or the pain) and a "disagreement point" effect (different family members may experience changes in their utility at the disagreement point in opposing directions).[15]

If utility is not Almost TU, it is not clear whether a change in the production possibility set caused by a change in government policy such as a change from individual taxation to joint taxation will benefit both spouses in the same direction when a GUBS is used to determine intrafamily distribution, even if the disagreement point is unaffected by the change in family taxation. In order to evaluate changes in family taxation, it is therefore useful to know whether GUBS plus Almost TU is a good approximation to household behavior.

Almost TU implies that a product mix is efficient independently of distribution. Yet empirical studies have found that a change in the disagreement point without changing the utility possibility set of spouses leads to a different household expenditure pattern or division of labor (e.g. Lundberg et. al, 1997, Rubaclava and Thomas, 2000, Chiappori et al., 2002). A change in expenditures on male vs. female clothing or male vs. female entertainment goods when interpreted as changes in expenditures on private consumption goods would be consistent with a model in which spouses have utilities over a household public good (corresponding to y_2 above) and disposable income (corresponding to y_1 above) that lead to Almost TU. A change in the division of labor, however, either refutes the assumption of Almost TU in the family bargaining context or suggests that more is going on than what is captured in a one period model.[16]

6.2 Incentive Compatibility

So far, we have considered exogenous changes in the production possibility set and assumed that agents produce efficiently the goods that they share according

[15] Gugl (2009) builds on that result in a two-period bargaining model. The model uses TU rather than Almost TU.

[16] See e.g. Gugl (2009) for a two-period model with TU that would imply a change in the division of labor if divorce laws change.

to a GUBS. Now suppose that agents choose their actions non-cooperatively to produce goods. In particular, denote by A_i the action set of an agent and by $a_i \in A_i$ a specific action taken by agent i. Denote by a_{-i} the actions taken by all the other agents except agent i. Each vector $a = (a_{i,} a_{-1}) \in A_i \times \prod_{j \neq i} A_j$ generates $y(a) \in Y\left(\prod_{i \in N} A_i\right)$.

Theorem (Incentive Compatibility) If agents use a $S \in \mathcal{G} \cup \mathcal{W}$ to determine the distribution of a given product mix $y(a)$, the unique subgame perfect Nash equilibrium (SPNE) outcome of the game in which agents sequentially choose their actions is efficient if and only if Almost TU holds.

Proof: Sufficiency: By Almost TU and Theorem 1. S satisfies solidarity. Therefore, all agents seek to maximize $\lambda(y(a), u)$. The fact that agents play a sequential game eliminates the possibility of a coordination problem and, hence, agents will non-cooperatively reach a vector of actions $a^* \in \arg\max_{a \in \prod_{i \in N} A_i} \lambda(y(a), u)$.

Necessity follows from Theorem 1: Only if Almost TU holds does a solution in $\mathcal{G} \cup \mathcal{W}$ guarantee that agents have a common goal (i.e., to maximize $\lambda(y(a), u)$).

6.2.1 Incentive Compatibility and Household Decision Making

The above model of spousal decision making also satisfies incentive compatibility in case of Almost TU: Both spouses have an incentive to provide the efficient amount of labor.

Corollary 1 *Suppose spouses agree to divide produced goods based on a GUBS but choose their labor supply individually. Each spouse chooses the efficient labor supply if and only if Almost TU holds.*

6.2.2 The Rotten Kid Theorem

Theorem 2 has also implications for Gary Becker (1974)'s Rotten Kid Theorem. Bergstrom (1989) formalizes the game that rotten kids play with their altruistic parent: In comparison with the model of spousal decision making introduced above, children now take the place of the spouses; each child's action impacts the production possibility set of the family. The parent in this game has a fixed amount of money at her disposal and, after observing her kids' actions, determines monetary transfers to its offspring by maximizing her altruistic utility function. Thus, the parent's altruistic preferences play the same role that the GUBS plays in the model of spousal decision making. Therefore, children thus take into account how the parent will react to their actions when they choose their own actions. The Rotten Kid Theorem states that even if the children are completely selfish and care only about their own consumption, they will behave as if they are maximizing the parent's altruistic utility function.

Bergstrom (1989)'s proof of the Rotten Kid Theorem requires TU, because he assumes that the parent treats every child's utility as a normal good in her altruistic utility function: Only if any action by a child, given the actions of all the other children results in a restricted utility possibility set in the form of a simplex are all the children guaranteed to benefit from taking efficient actions. In comparison to Bergstrom, we can weaken the requirement of TU to Almost TU by imposing a stronger, yet reasonable condition on the parent's altruistic utility function.

Corollary 2 *Suppose the parent's altruistic utility function takes on the form of a general utilitarian social welfare function. Each child behaves as if he/she would maximize the altruistic utility function of the parent if and only if children's utility functions lead to Almost TU.*

7 Conclusion

Many normatively appealing properties are also crucial in determining positive questions.[17] The solidarity property is no exception. We showed that for well-behaved General Utilatarian Bargaining Solutions the solidarity property is satisfied if and only if Almost TU holds. We then showed that if the agents can agree on how to distribute goods once they are produced, but choose their actions individually, incentive compatibility is satisfied if and only if the GUBS satisfies the solidarity property.

However, Almost TU is an important subdomain of all utility profiles and we believe Almost TU, combined with GUBS, to be a useful approach to modelling joint decisions in a variety of economic situations. We are also aware that one may take the opposite view, seeing our result as a damnation of GUBS because Almost TU seems to be rarely satisfied in practice. In that case the question becomes which class of bargaining solutions should take its place. The one that comes to mind immediately is the egalitarian solution (i.e., the solution that equally splits utility gains) or any other solution that plots a monotonic path through the disagreement point and pays no attention to the shape of the utility possibility set. It is obvious that such solutions satisfy the solidarity property and therefore incentive compatibility regardless of whether the utility profile leads to Almost TU or not.

However, such specifications are not without their own drawbacks. For example, consider a two stage game, in which agents can choose actions in the first stage that impact their disagreement utility as well as their joint production possibilities in the second stage. The second stage consists of the joint production and distribution of goods as modelled in this paper. The more bargaining solutions emphasize the disagreement point the more will agents inefficiently invest in their disagreement utility (Anbarci et al., 2002).

[17] See Moulin (1988) for a link between strong monotonicity in voting rules and strategy proofness.

8 References

Anbarci, N, S. Skaperdas and C. Syropoulos, 2002, Comparing Bargaining Solutions in the Shadow of Conflict: How Norms against Threats Can Have Real Effects, Journal of Economic Theory 106, 1-16.

Baland, J-M and J. A. Robinson, 2000, Is Child Labor Efficient? Journal of Political Economy 108, 663-79.

Becker, Gary S., 1974, A Theory of Social Interactions, Journal of Political Economy 82, 1063-94.

Bergstrom, Th. C., 1989, A Fresh Look at the Rotten Kid Theorem – and Other Household Mysteries, Journal of Political Economy 97, 1138-59.

Bergstrom, Th. C. and R. C. Cornes, 1983, Independence of Allocative Efficiency from Distribution in the Theory of Public Goods, Econometrica 51, 1753-66.

Bergstrom, Th. C. and R. C. Cornes, 1981, Gorman and Musgrave are Dual: An Antipodean Theorem on Public Goods, Economic Letters 7, 371-378.

Bergstrom, Th. C. and H. R. Varian, 1985, When do market games have transferable utility? Journal of Economic Theory 35, 222-33.

Bommier, A. and P. Dubois, 2004, Rotten Parents and Child Labor, Journal of Political Economy 112, 240-248.

Chiappori, Pierre-André, Bernard Fortin, and Guy Lacroix, 2002, Marriage Market, Divorce Legislation, and Household Labor Supply, Journal of Political Economy 110, 37-72.

Chun, Y. and W. Thomson, 1988, Monotonicity Properties of Bargaining Solutions When Applied to Economics" Mathematical Social Sciences 15, 11-27.

Gugl, Elisabeth, 2009, Income splitting, specialization, and intra-family distribution, Canadian Journal of Economics 42, 1050-71.

Kalai, E. and M. Smorodinsky, 1975, Other Solutions to Nash's Bargaining Problem, Econometrica 43, 513-18.

Lundberg, Shelly J., Robert A. Pollak and Terence J. Wales, 1997, Do Husbands and Wives Pool Their Resources? Evidence from the United Kingdom Child Benefit, Journal of Human Resources 32, 463-480.

Mas-Colell, A., M. D. Whinston and J. R. Green, 1995, Microeconomic Theory (Oxford University Press, New York).

Moulin, H., 1988, Axioms of Cooperative Decision Making (Cambridge University Press, Cambridge).

Nash, J. F. Jr., 1950, The Bargaining Problem, Econometrica 18, 155-62.

Pollak, R. A., 2006, Family Bargaining and Taxes: A Prolegomenon to the Analysis of Joint Taxation, Forthcoming in CESifo Economic Studies + MIT Press book

Rubalcava, Luis and Duncan Thomas, 2000, Family Bargaining and Welfare, Working Paper CCPR-007-00, California Center of Population Research.

Rubinstein, A., Z. Safra, and W. Thomson, 1992, On the Interpretation of the Nash Bargaining Solution and Its Extensions to Non-Expected Utility Preferences, Econometrica 60, 1171-81.

Sprumont, Y., 2008, Monotonicity and Solidarity Axioms in Economics and Game Theory, in *Rational Choice and Social Welfare*, Springer Berlin Heidelberg Eds. 71-94.

Xu, Y. and N. Yoshihara, 2008, The Behaviour of Solutions to Bargaining Problems on the Basis of Solidarity, *Japanese Economic Review* 59,133-138.

9 Appendix

9.1 Proof of Example 3

To find $(y,x) \in P(Y,u)$, first note that any efficient allocation must satisfy the following:

$$\begin{cases} y_1 = x_{11} + x_{12} \\ y_2 = x_{21} \\ y_3 = x_{32} \end{cases}$$

To find $P(Y,u)$ and $\partial U(Y,u)$ we solve:

$$\begin{aligned} \max_{x_{11},x_{12},x_{21},x_{32}} \quad & u(x_{11},x_{21}) \\ \text{s.t.} \quad & \psi_2 = u(x_{12},x_{32}) \\ & x_{11} + x_{12} + p_2 x_{21} + p_3 x_{32} = I \end{aligned}$$

This problem is equivalent to simultaneously

$$\begin{aligned} \max_{x_{11},x_{12}} \quad & u(x_{11},x_{21}) \\ \text{s.t.} \quad & x_{11} + x_{12} + p_2 x_{21} = I_1 \end{aligned}$$

and

$$\begin{aligned} \max_{x_{21},x_{32}} \quad & u(x_{12},x_{32}) \\ \text{s.t.} \quad & x_{11} + p_3 x_{32} = I_2 \end{aligned}$$

subject to

$$I_1 + I_2 = I.$$

It is obvious that the higher agent 1's share of I, the higher her utility. Thus each $\psi \in \partial U(Y,u)$ must be associated with a different distribution of I where agent 1 gains utility with every increase in her share and agent 2 loses utility with every decrease in his share. Homogeneity of degree one implies homotheticity which in turn implies that it is optimal for agent 1 to consume (x_{11},x_{21}) in the same proportion as before as her share of I increases. The same is true for agent 2 with respect to (x_{12},x_{32}) . By homogeneity of degree 1 of the utility functions, we also know that an increase in the share of I causes a proportional increase in u_i. Hence, the indirect utility function of agent 1 writes as follows

$$v(p_2,I_1) = \widetilde{v}(p_2) I_1$$

where

$$\begin{aligned} \widetilde{v}(p_2) &= \max_{x_{11},x_{21}} u(x_{11},x_{21}) \\ & \quad\text{s.t.} \quad x_{11}+p_2x_{21}=1 \end{aligned}$$

and we can write the indirect utility function of agent 2 as

$$v(p_3,I_2)=\widetilde{v}(p_3)\,I_2$$

where

$$\begin{aligned} \widetilde{v}(p_3) &= \max_{x_{12},x_{32}} u(x_{12},x_{32}) \\ & \quad\text{s.t.} \quad x_{12}+p_3x_{32}=1 \end{aligned}$$

Differentiating with respect to I_i:

$$\frac{\partial v_i}{\partial I_i}=\widetilde{v}(p_{i+1}).$$

This implies that as agent 1's share of I increases by one unit, her utility increases by $\widetilde{v}(p_2)$, and agent 2's utility decreases by $\widetilde{v}(p_3)$. Independent of how many units of I are already allocated to person 1, the decrease in person 2's utility and the increase in person 1's utility will always be the same as person 1 receives an additional unit of I. Therefore any (ψ_1,ψ_2) is found by

$$\psi_2=\lambda(Y,u_2)-\frac{\widetilde{v}(p_3)}{\widetilde{v}(p_2)}\psi_1.$$

Also note that

$$\frac{\widetilde{v}(p_3)}{\widetilde{v}(p_2)}=\frac{\widetilde{v}(p_3)\,I}{\widetilde{v}(p_2)\,I}=\frac{\lambda(Y,u_2)}{\lambda(Y,u_1)}.$$

Thus

$$\partial U(Y,u)=\left\{\psi\in\mathbb{R}_+^2 \middle| \frac{\psi_1}{\lambda(Y,u_1)}+\frac{\psi_2}{\lambda(Y,u_2)}=1\right\}.$$

Only in the special case in which $p_2=p_3=p$ such that $Y=\left\{y\in\mathbb{R}_+^3|y_1+p(y_2+y_3)=I\right\}$ we have

$$\lambda(Y,u_1)=\widetilde{v}(p)\,I=\lambda(Y,u_2).$$

Then

$$\partial U(Y,u)=\left\{\psi\in\mathbb{R}_+^2|\psi_1+\psi_2=\lambda(Y,u_1)=\lambda(Y,u_2)\right\}.$$