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On Periodic Structures and Testing for Seasonal Unit Roots*

Eric Ghysels[†], *Alastair Hall*[‡], *Hahn S. Lee*[+]

Abstract / Résumé

The standard testing procedures for seasonal unit roots developed so far have been based mainly on time invariant ARMA processes with AR polynomials involving seasonal differencing. One attractive alternative is to employ periodic ARMA models in which the coefficients are allowed to vary with the season. In this paper, we present convenient procedures for testing for the presence of unit roots at the zero and seasonal frequencies in periodic time series. The limiting distributions of these statistics are derived and tabulated. Simulation evidence illustrates the advantages of allowing for periodicity in this context when it is present. The tests are illustrated via applications to macroeconomic and ozone level data.

Les procédures standards pour tester la présence de racines unitaires aux fréquences saisonnières sont basées sur une représentation invariante ARIMA. Une classe alternative de processus est celle des modèles à variations périodiques des paramètres. Dans cette étude nous présentons des tests de racines unitaires qui prennent explicitement en compte une structure périodique. Les distributions asymptotiques sont dérivées. Une étude Monte Carlo démontre les avantages de nos tests par rapport aux procédures standards.

Key Words: periodic models, seasonal unit roots
Mots-clés : modèles périodiques, racines unitaires saisonnières

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1. INTRODUCTION

Two types of model specifications are most often considered for seasonal time series. One consists of time-invariant autoregressive integrated moving average (ARIMA) processes with AR polynomials involving first and/or seasonal differencing. This class of models, popularized through the work of Box and Jenkins (1976), has become standard textbook material.[1] A celebrated example of this class is the so-called airline model named after the passenger data set to which it was originally fitted. The second class has gained considerable interest in recent years, though it is still a distant second in terms of applications. Its original source of inspiration was the work of Gladysev (1961) on periodic autocorrelations and was later refined by Tiao and Grupe (1980). The models are usually referred to as periodic ARIMA models because they are characterized by deterministic seasonal variation in the parameters. Several papers explored the estimation and testing of periodic models, including Jones and Brelsford (1967), Pagano (1978), Troutman (1979), Tiao and Guttman (1980), Anděl (1983), Cipra (1983), Vecchia (1985a), Anděl (1987), Anděl (1989), Hurd and Gerr (1991), Lütkepohl (1991), Sakaï (1991), Vecchia and Ballerini (1991), Anderson and Vecchia (1993), Boswijk and Franses (1993), Ghysels and Hall (1993), McLeod (1993), Bentarzi and Hallin (1994), Franses (1994), among others. In addition, these models found successful applications in economics, environmental studies, hydrology and meteorology, see inter alia., Bhuiya (1971), Noakes et al. (1985), Vecchia (1985b), Vecchia et al. (1985), Osborn (1988), Birchenhall et al. (1989), Jiménez et al. (1989), Osborn and Smith (1989), Todd (1990), Ghysels and Hall (1992), McLeod (1993).

To date, tests of whether first or seasonal differencing is appropriate have been developed within the framework of time invariant ARIMA models; see inter alia, Hasza and Fuller (1982), Dickey, Hasza and Fuller (1984), Hylleberg et al. (1990). However, seasonal unit roots characterize the nonstationarity of periodic patterns in time series and so it is natural to test for these roots in the context of periodic models. In this paper, we propose a number of statistics which allow a researcher to test for the presence of zero and seasonal frequency unit roots in periodic AR models. We derive and tabulate the limiting distributions of our statistics. Simulation evidence

1 Besides textbooks, it is also worth mentioning survey papers on the subject such as Bell and Hillmer (1984) or Ghysels (1994). While the majority of the literature focuses on univariate models, some authors have studied multivariate extensions. Recent examples include Lee (1993) and Ahn and Reinsel (1994).

demonstrates that there can be considerable gains in power from taking account of the presence of periodicity when it is present.

An outline of the paper is as follows: in section 2, we examine the issue of testing for the presence of certain roots in the autoregressive polynomial of a periodic time series. Section 3 extends this analysis by introducing joint tests for the presence of these roots; one of these tests examines whether seasonal differencing is appropriate. Section 4 contains the results from a simulation study and an investigation of two empirical examples. All proofs are relegated to a mathematical appendix.

2. TESING FOR THE PRESENCE OF INDIVIDUAL ROOTS

Let the seasonal differencing operator to be defined as $\Delta_s = (1 - B^S)$ where B is the backshift operator and S is the seasonal sampling frequency. In the cases of annual, biannual, quarterly and monthly data, S takes the values 1, 2, 4, and 12, respectively. Following Box and Jenkins (1976), the seasonal differencing operator is applied to a series because it is believed to render a series stationary around, potentially, some deterministic level. However, although this transformation is a very natural choice, it actually amounts to an assumption about the values of roots of the autoregressive polynomial. For example:

$$\Delta = 1 - B$$

$$\Delta_2 = \Delta(1 + B)$$

$$\Delta_4 = \Delta_2 (1 + B^2)$$

$$\Delta_{12} = \Delta_4(1 + B + B^2)(1 - B + B^2)(1 + \sqrt{3}B + B^2)(1 - \sqrt{3}B + B^2).$$

Therefore, as is well-known, the use of Δ corresponds to the assumption of a real autoregressive root of 1; Δ_2 corresponds to real roots of ± 1; Δ_4 contains these two real roots plus the complex roots $\pm i$; Δ_{12} contains the roots of Δ_4 plus four additional pairs of complex conjugate roots. These roots imply different types of behavior. For example, the root of -1 corresponds to a component exhibiting two cycles per year and the roots of $\pm i$ correspond to a component exhibiting four cycles per year. From this perspective, it may be of interest to test for the presence of these individual effects. In this section, we develop test procedures that allow this in the context of periodic time series. In the next section, we extend this to joint tests which allow one, for instance, to test whether seasonal differencing is appropriate.

However, first we must address a matter of notation. In our presentation, it is necessary to distinguish the periodic function which determines a parameter value in a given period and the arguments of this function. All parameters are represented by "lower case" greek letters and we use ξ_t, say, to denote the periodic function $\sum_{s=1}^{S} D_{st}\,\xi_s$ where D_{st} is an indicator variable which takes the value 1 if $s = t \bmod S$. Similarly, ξ_{jt} denotes $\sum_{s=1}^{S} D_{st}\,\xi_{sj}$. It will always be clear from the context whether we refer to the function ξ_t or to the values it takes $\{\xi_s;\ s = 1, ..., S\}$.

It is most convenient to introduce the tests in the context of a zero mean periodic autoregressive model and then extend the results to models with an intercept and time trend. Consider the model:

$$y_t = \sum_{j=1}^{p} \rho_{jt}\, y_{t-j} + u_t. \tag{2.1}$$

Without loss of generality, we assume $t = (n - 1)S + s$ for $n = 1, 2, ..., N$ and $s = 1, 2, ..., S$; this gives a sample of size $T = NS$. To facilitate our analysis, we impose the following condition:

C.1: $\{u_t\}$ is a sequence of i.i.d. random variables with $E(u_t) = 0$, $E(u_t^2) = \sigma^2$ and $\sup_t E|u_t|^{\gamma} < \infty$ for some $\gamma > 2$.

Our inference is based on the regression models given in equations (2.2) and (2.3). First, consider the model:

$$y_t = \alpha_t\, y_{t-1} + \sum_{j=1}^{p-1} \theta_{jt}\, z^{\phi}_{1,t-j} + u_t \tag{2.2}$$

where $z^{\phi}_{1,t} = (1 - \phi B)y_t$. If y_t possesses a unit root at the zero frequency, then it has the representation in (2.2) with $\alpha_s = 1$, $s = 1, 2, ..., S$ and $\phi = 1$. If y_t has the root -1, then it has the representation in (2.2) with $\alpha_s = -1$, $s = 1, 2, ..., S$, and $\phi = -1$. Therefore, to test for the presence of either of these roots, one can estimate (2.2) with $\phi = c$ and test whether $\alpha_s = c$ for $c = \pm 1$. These two null hypotheses can be written compactly as:

$$H_0^R(\phi)\colon \alpha_s = \phi,\ s = 1, 2, ..., S$$

for $\phi = -1$ or 1; here the R superscript stands for "real" roots. The alternative denoted $H_1^R(\phi)$, is that at least one $\alpha_s \neq \phi$.

We now turn to inference about the complex roots. Consider the regression model:

$$y_t = \gamma_{1t}(-y_{t-1}) + \gamma_{2t}\, y_{t-2} + \sum_{j=1}^{p-2} \theta_{jt}\, z^{\phi}_{2,t-j} + u_t \tag{2.3}$$

where $z^{\phi}_{2,t} = (1 - \phi B + B^2)y_t$. Note that for notational convenience, the coefficients on $z^{\phi}_{1,t-j}$ and $z^{\phi}_{2,t-j}$ in equations (2.2) and (2.3) are both denoted θ_{jt}; however, the values taken by θ_{jt} are different in each case. This will not cause any ambiguity since none of the tests explicitly depend on θ_{jt}. If y_t possesses the complex conjugate pair of roots associated with $(1 - \phi z + z^2)$, then it has the representation in (2.3) with $\gamma_{1s} = \phi$, $\gamma_{2s} = 1$ for $s = 1, 2, ..., S$. Consequently, one can test for the presence of these roots by estimating (2.3) with the appropriate choice of ϕ in $z^{\phi}_{2,t}$ and testing if $\gamma_{1s} = \phi$, $\gamma_{2s} = 1$. This null hypothesis can be written compactly as:

$$H_0^C(\phi)\colon \gamma_{1s} = \phi,\ \gamma_{2s} = 1;\ s = 1, 2, ..., S$$

for $\phi = 0, \pm 1, \pm\sqrt{3}$. Here, the C superscript stands for "complex" roots. The alternative, denoted $H_1^C(\phi)$, is that at least one $\gamma_{1s} \neq \phi$ or one $\gamma_{2s} \neq 1$ in which case the series does not possess the roots associated with $(1 - \phi z + z^2)$.

All our inference procedures are based on the Wald statistic for testing linear restrictions on the parameters of a linear regression model estimated by ordinary least squares. The generic formula for the statistic is as follows. Suppose the regression model is:

$$y = X\beta + u$$

where y, u are $T \times 1$ vectors of observations on the dependent variable and error respectively; X is the $T \times k$ matrix of observations on the regressors. The Wald statistic for testing $R\beta = r$ is:

$$W = (R\hat{\beta} - r)' [R(X' X)^{-1}R']^{-1} (R\hat{\beta} - r) / \hat{\sigma}^2 \qquad (2.4)$$

where $\hat{\beta} = (X' X)^{-1} X' y$ and $\hat{\sigma}^2 = y'[I - X(X' X)^{-1} X']y / T$.

Let $W_S^R(\phi)$ denote the Wald statistic for testing $H_0^R(\phi)$ based on (2.2) and let $W_S^C(0)$ denote the Wald statistic for testing $H_0^C(\phi)$ based on (2.3). To present the limiting distribution of these statistics, we must introduce the following relation: let $B_S(r)$ denote an S-dimensional standard Brownian motion, G(r) denote the (4×1) standard Brownian motion given by:

$$G(r) = [S^{-1/2} G_1 (r), S^{-1/2} G_2 (r), (S/2)^{-1/2} G_3(r), (S/2)^{-1/2} G_4(r)]'$$

where $G_1(r) = \sum_{s=1}^{S} B_{Ss}(r)$, $G_2(r) = \sum_{s=1}^{S} (-1)^s B_{Ss}(r)$, $G_3(r) = \sum_{s=1}^{S/2} (-1)^{s-1} B_{Sj(s)}$,

$j(s) = 2s-1$, $G_4(r) = \sum_{s=1}^{S/2} (-1)^s B_{Sk(s)}$, $k(s) = 2s$. The distributions of these test statistics are as follows:

THEOREM 2.1: Let y_t be generated by (2.1) and assume C.1 and A.1 defined in the appendix hold, then: (i) under $H_0^R(\phi)$, $W_S^R(\phi) \Rightarrow \psi_S^R$, $\phi \pm 1$; (ii) under $H_0^C(\phi)$, $W_S^C(\phi) \Rightarrow \psi_S^C$, $\phi = 1, \pm 1, \pm\sqrt{3}$

where $\psi_S^R = \sum_{s=1}^{S} [\int_0^1 G_1(r)\, dB_{Ss}]^2 / \int_0^1 G_1(r)^2\, dr,$

$$\psi_S^C = \text{trace} \{\int_0^1 G_{34}(r)'\, dG(r) [\int_0^1 G_{34}(r)\, G_{34}(r)'\, dr]^{-1} \int_0^1 G_{34}(r)\, dG(r)'\},$$

and $G_{34}(r)$ is the (2×1) subvector of $G(r)$ containing its 3rd and 4th elements.

The limiting distributions only depend on the known parameter S. Percentiles are presented in Table 2.1 for S = 4,12.[2] The table covers the case without intercept and linear trend. The intercept case, as well as intercept plus trend cases, are discussed next.

In many cases, it may indeed be appropriate to include an intercept or time trend in the model. Accordingly, consider the models:

$$y_t = \alpha_t\, y_{t-1} + \mu_t + \sum_{j=1}^{p-1} \theta_{jt}\, z^{\phi}_{1,t-j} + u_t, \tag{2.5}$$

$$y_t = \alpha_t\, y_{t-1} + \mu_t + \beta_t(n - N/2) + \sum_{j=1}^{p-1} \theta_{jt}\, z^{\phi}_{1,t-j} + u_t, \tag{2.6}$$

$$y_t = \gamma_{1t}(-y_{t-1}) + \gamma_{2t}\, y_{t-2} + \mu_t + \sum_{j=1}^{p-2} \theta_{jt}\, z^{\phi}_{2,t-j} + u_t, \tag{2.7}$$

$$y_t = \gamma_{1t}(-y_{t-1}) + \gamma_{2t}\, y_{t-2} + \mu_t + \beta_t(n - N/2) + \sum_{j=1}^{p-2} \theta_{jt}\, z^{\phi}_{2,t-j} + u_t, \tag{2.8}$$

Let $W^R_{S\mu}(\phi)$, $W^R_{S\tau}(\phi)$ be the Wald statistics for testing $H^R_0(\phi)$ based on (2.5) and (2.6), respectively. Likewise, let $W^C_{S\mu}(\phi)$, $W^C_{S\tau}(\phi)$ be the Wald statistics for testing $H^C_0(\phi)$ based on (2.7) and (2.8), respectively. The limiting distributions of these statistics are as follows:

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2 All computations were performed using the RATS, Version 4.01, package of ESTIMA, Inc. To calculate the critical values, we used 10,000 iterations. For S = 12 and N = 20, we only report the case of no intercept and trend since the other cases yielded essentially similar critical values.

THEOREM 2.2: Let y_t be generated by (2.1) assume C.1 and assumption A.1 defined in the appendix hold, then: (i) under $H_0^R(\phi)$, $W_{S\mu}^R(\phi) \Rightarrow \psi_{S\mu}^R$, $W_{S\tau}^R(\phi) \Rightarrow \psi_{S\tau}^R$ for $\phi = \pm 1$; (ii) under $H_0^C(\phi)$: $W_{S\mu}^C(\phi) \Rightarrow \psi_{S\mu}^C$, $W_{S\tau}^C(\phi) \Rightarrow \psi_{S\tau}^C$ for $\phi = 0, \pm 1, \pm\sqrt{3}$.

For brevity, these limiting distributions are defined in the appendix; again, they only depend on S and percentiles are presented in Table 2.1 as noted before.

Finally, we observe that the statistics $W_S^R(\phi)$ are asymptotically equivalent to the sum over s = 1, 2, ..., S of the squared t-statistics for H_o: $\alpha_s = \phi$ from the appropriate regression model. This provides a convenient method of calculation from standard regression computer output.

3. TESTING FOR SEASONAL DIFFERENCING

We now turn to the question of testing the hypothesis that seasonal differencing would yield a stationary series. From the previous section, it is clear that this amounts to testing a joint hypothesis about the roots of the autoregressive polynomial. To illustrate the structure of these joint tests, we concentrate on the case where S = 4. The procedures easily extend to the case where S = 12 and this is discussed in the appendix. Let $y_{1t} = (1 + B + B^2 + B^3)y_t$, $y_{2t} = -(1 - B + B^2 - B^3)y_t$, $y_{3t} = -(1 + B^2)y_t$, $z_t^4 = (1 - B^4)y_t$ and consider the regression model:

$$z_t^4 = \pi_{1t}\, y_{1t-1} + \pi_{2t}\, y_{2t-1} + \pi_{3t}\, y_{3t-1} + \pi_{4t}\, y_{3t-2} + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^4 + u_t. \quad (3.1)$$

In the context of aperiodic time series, Hylleberg et al. (1990) showed that various parameter restrictions among the π coefficients correspond to the existence of the roots discussed in the previous section. Ghysels, Lee and Noh (1994) showed that this procedure can be extended to test for seasonal differencing. In this section, we generalize this framework to periodic time series.

If y_t possesses all the roots ± 1, $\pm i$, then it has the representation in (3.1) with $\pi_{is} = 0$, i, s = 1, 2, 3, 4. This corresponds to the case where seasonal differencing yields stationarity.

Table 2.1: Percentiles of Wald Statistics for Real and Pair of Complex Roots

Quaterly Case (S = 4)

	1%	5%	10%	25%	50%	75%	90%	95%	99%
Sample size 100 years (N=100, T=400)									
W^R_4	1.810	3.343	4.293	6.335	8.968	12.165	15.817	18.306	23.061
$W^R_{4\mu}$	2.688	4.554	5.678	7.930	10.840	14.288	17.951	20.372	25.810
$W^R_{4\tau}$	3.666	5.884	7.242	9.623	12.881	16.708	20.516	23.245	29.073
W^C_4	6.722	8.946	10.406	13.228	16.866	21.100	25.467	28.237	34.787
$W^C_{4\mu}$	8.990	11.689	13.425	16.498	20.609	25.442	30.364	33.333	39.415
$W^C_{4\tau}$	11.745	15.020	16.870	20.385	24.767	29.951	35.088	38.168	44.741
Sample size 50 years (N = 50, T = 200)									
W^R_4	1.689	3.198	4.206	6.066	8.651	11.837	15.380	17.916	22.708
$W^R_{4\mu}$	2.431	4.323	5.425	7.626	10.384	13.926	17.705	20.365	26.092
$W^R_{4\tau}$	3.577	5.560	6.804	9.110	12.233	16.073	19.973	22.556	27.912
W^C_4	6.307	8.610	10.010	12.837	16.507	20.720	25.093	28.069	34.340
$W^C_{4\mu}$	8.511	11.195	12.785	15.945	20.130	24.774	29.711	32.779	39.481
$W^C_{4\tau}$	11.120	14.138	16.087	19.530	23.936	29.063	34.345	37.579	45.164
Sample size 20 years (N = 20, T = 80)									
W^R_4	1.492	2.848	3.672	5.403	7.937	11.027	14.693	17.159	22.541
$W^R_{4\mu}$	2.114	3.627	4.553	6.543	9.170	12.514	16.450	19.248	24.796
$W^R_{4\tau}$	2.837	4.517	5.488	7.599	10.467	14.088	18.240	21.179	28.275
W^C_4	5.551	7.665	9.021	11.652	15.232	19.706	24.736	28.050	35.294
$W^C_{4\mu}$	7.005	9.747	11.289	14.224	18.289	23.205	28.657	32.423	40.046
$W^C_{4\tau}$	8.709	11.657	13.383	16.804	21.260	26.867	32.639	36.430	45.999

Table 2.1 continued: Percentiles of Wald Statistics for Real and Pair of Complex Roots

Monthly Case (S = 12)

	1%	5%	10%	25%	50%	75%	90%	95%	99%
				Sample size 100 years (N=100, T=1,200)					
W^R_{12}	8.474	12.762	15.266	19.946	26.193	33.161	40.440	45.177	54.393
$W^R_{12\mu}$	9.173	13.501	15.955	20.934	27.193	34.652	42.104	46.894	56.703
$W^R_{12\tau}$	9.586	14.122	16.739	21.818	28.362	35.969	43.477	48.485	58.392
W^C_{12}	42.910	50.301	54.191	61.112	69.323	78.017	86.728	92.298	103.304
$W^C_{12\mu}$	46.921	54.778	59.072	66.360	74.920	84.044	92.721	98.845	109.956
$W^C_{12\tau}$	51.162	59.907	64.169	71.688	80.663	89.818	99.485	105.676	117.057
				Sample size 50 years (N = 50,T = 600)					
W^R_{12}	7.396	110.392	12.293	15.873	20.677	27.113	34.274	39.382	50.052
$W^R_{12\mu}$	8.012	11.181	13.117	16.715	21.722	27.920	34.994	39.836	51.684
$W^R_{12\tau}$	9.111	12.265	14.163	17.777	22.745	29.050	35.716	40.231	51.027
W^C_{12}	38.137	44.470	48.312	54.787	62.604	71.078	79.854	85.189	95.497
$W^C_{12\mu}$	41.231	48.113	52.049	58.625	66.917	76.100	84.630	90.279	100.607
$W^C_{12\tau}$	44.679	51.937	55.926	62.990	71.577	80.826	89.782	95.617	107.180
				Sample size 20 years (N = 20, T = 240)					
W^R_{12}	3.733	5.728	6.985	9.482	13.272	18.052	23.566	27.448	35.635
W^C_{12}	12.057	16.205	18.759	23.330	29.444	37.041	45.334	51.098	63.915

Note: For a definition of the tests, see Theorems 2.1 and 2.2.

We denote this null hypothesis by:

$H_0^A(4)$: $\pi_{is} = 0$ for all i, s = 1, 2, ..., 4

where the A superscript stands for "all roots" and the 4 refers to the quarterly data. The alternative, $H_1^A(4)$ is that at least one $\pi_{is} \neq 0$.

A related hypothesis is whether all the "seasonal roots" $-1, \pm i$ are present. If this is the case, then y_t has the representation in (3.1) with $\pi_{is} = 0$ for $i = 2, 3, 4$, $s = 1, \ldots, 4$. Note that this representation is valid irrespective of whether y_t possesses the root 1, i.e., a unit root at the zero frequency. We denote this null hypothesis by:

$H_0^S(4)$: $\pi_{is} = 0$ $i = 2, 3, 4$, $s = 1, 2, \ldots, 4$

where the S superscript stands for "seasonal roots"; again the alternative is that $\pi_{is} \neq 0$ for at least one $i > 1$ and one s.

Let W_S^A, W_S^S with S = 4 denote the Wald statistics for testing $H_0^A(4)$ and $H_0^S(4)$, respectively. The limiting distributions of these statistics are derived in the appendix. The notation for these distributions is presented in Table 3.1 and the percentiles are given in Table 3.2. One may also wish to include an intercept or a time trend in the model and so estimate either:

$$z_t^4 = \sum_{i=1}^{3} \pi_{it}\, y_{i,t-1} + \pi_{4t}\, y_{3,t-2} + \mu_t + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^4 + u_t \tag{3.2}$$

or

$$z_t^4 = \sum_{i=1}^{3} \pi_{it}\, y_{i,t-1} + \pi_{4t}\, y_{3,t-2} + \mu_t + \beta_t(n - N/2) + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^4 + u_t. \tag{3.3}$$

The presence of the deterministic terms in (3.2) and (3.3) does not alter the arguments above, although it does change the limiting distributions. Let $W_{S\mu}^A$, $W_{S\tau}^A$ with $S = 4$ be the Wald statistics for testing $H_0^A(4)$ based on (3.2) and (3.3), respectively. Similarly, let $W_{S\mu}^S$, $W_{S\tau}^S$ be the Wald statistics for testing $H_0^A(4)$ based on (3.2) and (3.3). The limiting distributions are summarized in Table 3.1 and described in the appendix.

For the case where S = 12, one must modify the regression models in the fashion shown in the appendix. The notation for these tests is analogous to the quarterly case:

$H_0^A(12)$: $\Delta_{12}\, y_t$ is stationary

$H_0^S(12)$: y_t possesses the roots of Δ_{12} / Δ,

and W_{12}^A is the Wald test of $H_0^A(12)$ based on the monthly analogs of (3.1) (equation (A.21) in the appendix), etc. The limiting distributions are summarized in Table 3.1 and the percentiles presented in Table 3.2.

We conclude this section by noting that all the limiting distributions presented in this section are free of nuisance parameters.

Table 3.1: Test Statistics and Their Limiting Distributions

Null hypothesis	Regression model	Limiting distributions of Wald statistics
$H_0^A(4)$	(3.1)	ψ_4^A
	(3.2)	$\psi_{4\mu}^A$
	(3.3)	$\psi_{4\tau}^A$
$H_0^S(4)$	(3.1)	ψ_4^S
	(3.2)	$\psi_{4\mu}^S$
	(3.3)	$\psi_{4\tau}^S$
$H_0^A(12)$	(A.21)	ψ_{12}^A
	(A.22)	$\psi_{12\mu}^A$
	(A.23)	$\psi_{12\tau}^A$
$H_0^S(12)$	(A.21)	ψ_{12}^S
	(A.22)	$\psi_{12\mu}^S$
	(A.23)	$\psi_{12\tau}^S$

Table 3.2: Percentiles of Wald Statistics for Seasonal Differencing and Seasonal Unit Roots

Quaterly Case (S = 4)

	1%	5%	10%	25%	50%	75%	90%	95%	99%
				Sample size 100 years (N = 100, T = 400)					
W_4^A	14.522	17.669	19.444	22.890	27.171	32.055	37.119	40.615	47.400
$W_{4\mu}^A$	20.302	24.038	26.140	29.968	35.034	40.702	46.047	49.823	57.531
$W_{4\tau}^A$	26.814	31.004	33.600	38.225	43.695	49.983	56.210	60.319	68.166
W_4^S	10.944	13.712	15.503	18.710	22.889	27.620	32.406	35.719	42.412
$W_{4\mu}^S$	14.883	18.174	20.322	24.003	28.773	34.216	39.524	42.910	50.215
$W_{4\tau}^S$	19.300	23.239	25.623	29.868	35.125	40.937	46.908	50.853	58.263
				Sample size 50 years (N = 50,T = 200)					
W_4^A	14.354	17.248	19.037	22.508	26.778	31.902	37.261	40.792	47.674
$W_{4\mu}^A$	20.101	23.422	25.588	29.510	34.621	40.495	46.119	50.104	58.108
$W_{4\tau}^A$	26.204	30.238	32.729	37.410	43.200	49.960	56.613	61.095	70.105
W_4^S	10.720	13.478	15.120	18.350	22.489	27.361	32.349	35.671	42.264
$W_{4\mu}^S$	14.678	17.655	19.626	23.404	28.230	33.804	39.268	42.796	50.220
$W_{4\tau}^S$	18.848	22.506	24.700	28.947	34.298	40.333	46.497	50.441	59.398
				Sample size 20 years (N = 20, T = 80)					
W_4^A	13.076	16.078	17.932	21.423	26.113	32.059	38.280	42.877	51.451
$W_{4\mu}^A$	17.738	21.420	23.757	27.910	33.448	40.426	47.742	52.722	63.619
$W_{4\tau}^A$	22.582	27.080	29.665	34.670	41.373	49.341	58.006	64.284	77.639
W_4^S	9.945	12.409	14.009	17.239	21.515	26.836	32.500	36.605	45.004
$W_{4\mu}^S$	12.630	15.932	17.820	21.528	26.435	32.593	39.443	43.999	53.143
$W_{4\tau}^S$	15.723	19.496	21.724	26.156	31.782	38.695	46.020	51.514	62.711

Table 3.2 continued: Percentiles of Wald Statistics for Seasonal Differencing and Seasonal Unit Roots
Monthly Case (S = 12)

	1%	5%	10%	25%	50%	75%	90%	95%	99%
					Sample size 100 years (N = 100, T = 1,200)				
W^A_{12}	91.958	101.439	106.603	115.888	126.880	138.168	149.116	156.248	170.372
$W^A_{12\mu}$	99.981	110.204	115.000	126.725	137.725	149.804	160.734	167.567	182.147
$W^A_{12\tau}$	109.133	120.448	126.126	136.579	148.460	160.830	173.004	180.182	194.198
W^S_{12}	68.810	77.937	82.607	91.009	100.975	111.590	121.775	128.167	141.655
$W^S_{12\mu}$	75.265	85.081	90.221	99.166	109.625	120.673	131.214	137.979	151.119
$W^S_{12\tau}$	82.350	92.858	98.492	107.435	118.595	129.969	140.906	147.913	160.784
					Sample size 50 years (N = 50,T = 600)				
W^A_{12}	82.660	92.212	97.135	106.478	117.424	129.481	140.894	147.962	161.876
$W^A_{12\mu}$	90.320	99.809	105.257	114.731	126.302	138.975	150.908	158.136	172.218
$W^A_{12\tau}$	97.161	107.502	113.627	123.693	135.548	148.666	161.153	168.623	184.404
W^S_{12}	61.650	69.798	74.387	82.947	92.696	103.100	113.270	119.693	131.153
$W^S_{12\mu}$	66.431	75.841	80.789	89.188	99.537	110.628	121.558	128.125	140.958
$W^S_{12\tau}$	72.934	82.134	87.076	96.243	106.931	118.111	129.534	136.389	149.963
					Sample size 20 years (N = 20, T = 240)				
W^A_{12}	31.684	39.079	43.390	51.211	61.652	73.871	87.401	96.329	116.762
W^S_{12}	22.448	28.461	31.700	37.951	46.315	56.534	67.421	75.271	91.693

Note: See Table 3.1 for definitions of test statistics.

4. SIMULATION EVIDENCE OF FINITE SAMPLE PROPERTIES AND EMPIRICAL APPLICATIONS

In this final section, we report results of a Monte Carlo study of the finite sample properties of the statistics presented in the previous two sections and then two empirical applications.

The design of the experiments was based on the following data generating process:

$$(1 - a_s B)\,(1 + B)\; 1 - a_s B^2)\; y_t = u_t \tag{4.1}$$

where u_t is i.i.d. N(0,1) and $t = (n - 1)\,4 + s$. Notice, we focus exclusively on a quarterly model where periodic behavior may appear at the zero and seasonal frequencies; the values of a_s are given in Table 4.1. It should be noted that a_s was selected to control both types of roots simultaneously in order to keep the number of cases limited. A total of six test statistics were considered, three of which are commonly used and do not explicitly exploit the periodic features in the DGP, and three statistics introduced in sections 2 and 3. The first set of statistics includes: (a) the Dickey-Fuller t statistics, denoted DF; (b) the joint test proposed by Ghysels, Lee and Noh (1994) for the presence of unit roots at all the seasonal frequencies, denoted GLN; and (c) the joint test for the $(1 - B^4)$ operator proposed by Hylleberg et al. (1990), denoted HEGY. In each case, the auxiliary regression models did not include a trend nor seasonal dummies or a constant. The sample size selected was 20 years, or 80 observations. The second set of three statistics includes: (a) the $W_4^R(1)$ statistic described in Theorem 2.1, (b) the W_4^S statistic, and (c) the W_4^A statistic both appearing in section 3. Hence, the first and second set of test statistics cover similar hypotheses regarding the presence unit roots at the zero and seasonal frequencies.

Table 4.1 reports simulation results based on 10,000 Monte Carlo simulation using the RNDN function of the GAUSS package. The top line of Table 4.1 shows that none of the statistics show any noticable size distortion. The next line in Table 4.1 stresses an interesting feature as it relates to a case where the product of the α_s coefficients equals one, yet with the α_s differing dramatically. Let us first focus on the first set of three statistics. First, we notice that the DF statistic has its power equal to its size while the two joint statistics GLN and HEGY reject the null outright.

Table 4.1: Monte Carlo Design and Results

a_1	a_2	a_3	a_4	$\prod_{s=1}^{4} a_s$	DF		GLN		HEGY		$W_4^R(1)$		W_4^S		W_4^A	
					5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
1.00	1.00	1.00	1.00	1.00	0.04	0.09	0.05	0.10	0.06	0.11	0.04	0.10	0.05	0.10	0.04	0.09
0.80	1.25	0.80	1.25	1.00	0.04	0.09	1.00	1.00	1.00	1.00	0.48	0.63	0.99	1.00	0.98	0.99
1.00	1.00	1.00	-1.00	-1.00	0.79	0.81	0.79	0.81	0.80	0.82	0.04	0.09	1.00	1.00	1.00	1.00
1.00	0.80	1.00	0.80	0.64	0.35	0.54	0.12	0.20	0.32	0.46	0.14	0.24	0.03	0.07	0.12	0.22

Notes: DF: Dickey–Fuller t statistics; GLN: statistics for roots at seasonal frequencies in Ghysels, Lee and Noh; HEGY: statistic for seasonal differencing in Hylleberg et al. All three remaining statistics are defined in sections 2 and 3. All computations involved 10,000 iterations. Sample size is 20 years; DGP is described by equation (4.1).

This first case stresses the advantage of taking periodicity into account as is done in the second block of three statistics. Indeed, with the product of the a_s coefficients equal to one, the DF statistic is "tricked" by the fact that, on average across all four seasons, there is a unit root. The GLN and HEGY statistics are not affected by the fact that $\prod_{s=1}^{4} a_s = 1$, instead they would be affected by for instance $\prod_{s=1}^{4} (a_s)^{1/2} = 1$. Looking at the three statistics together, DF, GLN and HEGY, one would conclude in most circumstances that one should take a first difference of the data. Instead, the periodic tests, $W_4^R(1)$, W_4^S and W_4^A, show good power properties in rejecting unit root behavior at all the frequencies. The next case is also particularly interesting. The product of the a_s coefficients now equals -1, because all but one coefficient equal 1.0 and the fourth is -1. Let us first discuss what impact this has on the data generating process appearing in (4.1). Since the polynomial on the left-hand side equals $(1 - a_sB)(1 + B)(1 + a_sB^2)$, one finds for the three seasons $(1 - B)(1 + B)(1 + B^2)$ while for the fourth season, the polynomial equals $(1 + B)^3 (1 - B)$. Hence, in each of the four seasons, the polynomial contains the $(1 - B)$ unit root. Yet, looking at the results in Table 4.1, we notice that the DF statistic strongly rejects the zero frequency unit root hypothesis, simply because $\prod_{s=1}^{4} a_s = -1$ and no unit root behavior is detected on average. In contrast, the $W_4^R(1)$ statistic correctly identifies the zero frequency unit root while the W_4^S and W_4^A also strongly reject the presence of unit roots at all seasonal frequencies. The final case appearing in Table 4.1 stresses the fact that the nonperiodic tests may be powerful, nevertheless. Here, the product of the a_s coefficients equals 0.64 which is far from the unit circle yet two coefficients equal to 1.0 while the two others equal 0.8. Comparing DF, GLN and HEGY with the periodic tests reveals that the former group of tests is more powerful in these circumstances. Such a DGP is probably uncommon in practice yet it is useful here to point out situations where traditional tests are more powerful.

To conclude, we consider some empirical applications which draw upon Osborn (1988), Osborn and Smith (1989) and Bloomfield, Hurd and Lund (1994). The former two applied periodic models to economic time series while the latter studied stratospheric ozone data with similar models. Using the data from the original articles, we apply our tests as well as the three nonperiodic tests considered in the Monte Carlo simulations. Osborn and Smith (1989) examine U.K. quarterly consumers' expenditures and assess the benefits that may accrue from the use of periodic models. Nondurable consumer goods are available in a number of categories: alcoholic drink and tobacco; clothing, footwear; and energy products. To this set of series, we also add the total of nondurable consumption as well as disposable income and prices [the latter are studied in Osborn (1988)]. All data cover a sample from 1955:1 until 1984:2. The results appearing in the top panel, covering the quarterly data series, underline the benefits of allowing for periodicity in testing for unit roots in seasonal data. With the GLN and HEGY test statistics, one would accept the presence of unit roots at seasonal frequencies in several cases. In contrast, for none of the eight series is there supporting evidence of unit roots at seasonal frequencies according to the W_4^S and W_4^A statistics. For the zero frequency unit root, the results are more mixed, often finding agreement between the DF and $W_S^R(1)$ statistics.

Table 4.2: Empirical Results of Tests for Unit Roots in Periodic Time Series

Data	DF	GLN	HEGY	$W_S^R(1)$	W_S	W_S^A
<u>Quarterly S = 4</u>						
U.K. Income	3.76**	14.28**	14.09**	18.69**	83.02**	100.63**
U.K. Nondurables	2.68*	0.29	2.00	20.33**	40.50**	50.71**
Prices	1.65	23.88**	18.38**	8.48	195.33**	199.36**
Food	3.34**	2.59*	4.95**	18.06**	50.54**	64.77**
Alcohol	3.35**	0.27	3.03**	11.90	49.56**	63.26**
Footwear	2.19	1.30	2.23	19.73**	62.77**	72.02**
Clothing	2.85	0.14	2.17	8.63	42.98**	55.46**
Energy	4.73**	5.70**	10.06**	6.31	34.70*	64.60**
<u>Monthly S = 12</u>						
Arosa Stratospheric Ozone Data	5.81**	31.81**	31.93**	20.06	985.88**	1007.4**

Notes: For description test statistics, see Table 4.1. The quarterly data are taken from Osborn (1988) and Osborn and Smith (1989). The monthly data are from Bloomfield, Hurd and Lund (1994).

A second and final data set contains 50 years of monthly observations of stratospheric ozone data from Arosa, Switzerland. Bloomfield, Hurd and Lund show that the correlation structure of such data displays strong periodic features and suggests an ARMA model with periodically varying coefficients to fit the data. According to the results appearing in Table 4.2, we find one significant difference between the left and right panels, respectively, covering tests based on nonperiodic and periodic models. Indeed, we find that the zero frequency unit root hypothesis cannot be rejected with the $W_S^R(1)$ test. This appears to contradict the evidence based on a standard DF test.

APPENDIX A

We first present some useful notations and results which will be used below to develop the asymptotic distribution theory for the statistics proposed in the text.

Define:

$$w_{kn} = \sum_{i=1}^{n} \sum_{s=1}^{S} e_{ki}(s) \qquad \text{for } k = 1, 2, 3, 4, \tag{A.1}$$

where

$$e_{1i}(s) = u_{(i-1)S+s}, \qquad e_{2i}(s) = (-1)^s u_{(i-1)S+s},$$

$$e_{3i}(s) = \sin\tfrac{\pi}{2}[(i-1)S+s]\, u_{(i-1)S+s} \text{ and}$$

$$e_{4i}(s) = \cos\tfrac{\pi}{2}[(i-1)S+s]\, u_{(i-1)S+s}.$$

Note that (A.1) implies that:

$$w_{1n} = \sum_{t=1}^{nS} u_t, \quad w_{2n} = \sum_{t=1}^{nS} (-1)^t u_t$$

$$w_{3n} = \sum_{t=1}^{nS} \sin(\tfrac{\pi}{2}t)\, u_t, \text{ and } w_{4n} = \sum_{t=1}^{nS} \cos(\tfrac{\pi}{2}t)\, u_t.$$

Note also that:

$$w_{kn} = w_{k,n-1} + v_{kn} = \sum_{i=1}^{n} v_{ki} \tag{A.2}$$

where $v_{kn} = \sum_{s=1}^{S} e_{kn}(s)$.

Let U_{kn} denote $S \times 1$ vectors such that:

$$U_{kn} = [e_{kn}(1), e_{kn}(2), \ldots, e_{kn}(S)]' \qquad \text{for } k = 1, 2, 3, 4. \tag{A.3}$$

From Phillips and Durlauf (1986, Theorem 2.1), we have:

$$N^{-1/2} \sum_{n=1}^{[Nr]} U_{1n} \to \sigma B_1(r) \equiv \sigma W(r) \tag{A.4.1}$$

where $B_1(r) \equiv W(r)$ is an S-dimensional standard Brownian motion with s^{th} element $W_s(r)$. Similarly, we can show that:

$$N^{-1/2} \sum_{n=1}^{[Nr]} U_{2n} \to \sigma B_2(r) \tag{A.4.2}$$

where $B_2(r)$ is an S-dimensional standard Brownian motion with s^{th} element $B_{2s}(r) = (-1)^s$ $W_s(r)$ for $s = 1, ..., S$.

Noting that:

$e_{3n}(s) = 0 \quad$ for $S = 2k + 2$ and $k = 0, 1, ...$

$e_{3n}(s) = u_{(n-1)S+s} \quad$ for $s = 4k + 1$

$e_{3n}(s) = (-1)\, u_{(n-1)S+s} \quad$ for $s = 4k + 3$

it can be shown that $N^{-1/2} \sum_{n=1}^{[Nr]} U_{3n} \to \sigma B_3(r)$, where

$$B_3(r) = [W_1(r), 0, -W_3(r), 0, ...]. \tag{A.4.3}$$

From the definition of $e_{4n}(s)$, we can similarly show that $N^{-1/2} \sum_{n=1}^{[Nr]} U_{4n} \to \sigma B_4(r)$, where

$$B_4(r) = [0, -W_2(r), 0, W_4(r), ...]. \tag{A.4.4}$$

Using (A.2), we have:

$$w_{kN} = \sum_{n=1}^{N} v_{kn} = \iota' \sum_{n=1}^{N} U_{kn}$$

where ι is an S-dimensional vector of ones. From the relations in (A.4), it follows that:

$$N^{-1/2} \sum_{n=1}^{[Nr]} v_{kn} \to \sigma G_k(r) \qquad \text{(A.5)}$$

where $G_k(r) = \sum_{j=1}^{S} B_{kj}(r)$. Note here that from the relations (A.4.1)-(A.4.4), we have $G_1(r) = \sum_{j=1}^{S} W_j(r)$, $G_2(r) = \sum_{j=1}^{S} (-1)^i W_j(r)$, $G_3(r) = \sum_{j=1}^{S/2} (-1)^{j-1} W_{2j-1}(r)$, and $G_4(r) = \sum_{j=1}^{S/2} (-1)^j W_{2j}(r)$.

Finally, let y_{kt} (k = 1, 2, 3) denote the time series processes generated by the following equations:

$$y_{1t} = y_{1,t-1} + \sum_{j=1}^{p-1} \theta_{tj} z^{1}_{1,t-j} + u_t \qquad \text{(A.6.1)}$$

$$y_{2t} = -y_{2,t-1} + \sum_{j=1}^{p-1} \theta_{tj} z^{-1}_{1,t-j} + u_t \qquad \text{(A.6.2)}$$

$$y_{3t} = -y_{3,t-2} + \sum_{j=1}^{p-2} \theta_{tj} z^{0}_{2,t-j} + u_t. \qquad \text{(A.6.3)}$$

The processes z^{ϕ}_{kt} are defined following equations (2.2) and (2.3) for k = 1 and k = 2, respectively. Furthermore, we shall assume the following:

Assumption A.1: The z_{kt}^{ϕ} processes have an infinite order moving average representation

$$C(B)u_t = \sum_{i=0}^{\infty} C_i\, u_{t-i}, \text{ where } \sum_{i=0}^{\infty} i|C_i| < \infty. \tag{A.7}$$

The following relations are useful in deriving the asymptotic distribution of the test statistics in Theorem 2.1.

Lemma A.1: As $T \to \infty$ (and thus $N \to \infty$), we have:

(i) $$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{1,t-1}^2 \to C(1)^2\, \sigma^2 \int_0^1 G_1(r)^2\, dr \tag{A.8.1}$$

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{2,t-1}^2 \to C(-1)^2\, \sigma^2 \int_0^1 G_2(r)^2\, dr \tag{A.8.2}$$

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{3,t-1-i}^2 \to [\sin\tfrac{\pi}{2}(s-i)]^2\, \sigma^2\, [C_R^2 \int_0^1 G_4(r)^2\, dr$$

$$+ C_I^2 \int_0^1 G_3(r)^2\, dr - C_R\, C_I \int_0^1 G_4(r)\, G_3(r)\, dr + [\cos\tfrac{\pi}{2}(s-i)]^2\, \sigma^2\, [C_R^2 \int_0^1 G_3(r)^2\, dr$$

$$+ C_I^2 \int_0^1 G_4(r)^2\, dr + C_R\, C_I \int_0^1 G_3(r)\, G_4(r)\, dr] \quad \text{for } i = 0, 1 \tag{A.8.3}$$

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{3,t-1}\, y_{3,t-2} \to (-1)^s\, \sigma^2\, [(C_R^2 - C_I^2) \int_0^1 G_3(r)^2\, G_4(r)\, dr$$

$$+ C_R\, C_I \int_0^1 (G_3(r)^2 - G_4(r)^2)\, dr] \tag{A.8.4}$$

(ii) $$N^{-1} \sum_{t=1}^{T} D_{st}\, y_{1,t-1}\, u_t \to C(1)\, \sigma \int_0^1 G_1(r)\, dB_{1s}(r) \tag{A.9.1}$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{2,t-1})\, u_t \to C(-1)\, \sigma \int_0^1 G_2(r)\, dB_{2s}(r) \tag{A.9.2}$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{3,t-1})\, u_t \rightarrow \cos(\tfrac{\pi}{2} s)\, \sigma\, [C_R \int_0^1 G_3(r)\, dB_{4s}(r) + C_I \int_0^1 G_4(r)\, dB_{4s}(r)]$$

$$- \sin(\tfrac{\pi}{2} s)\, \sigma\, [C_R \int_0^1 G_4(r)\, dB_{3s}(r) - C_I \int_0^1 G_3(r)\, dB_{3s}(r)] \qquad \text{(A.9.3)}$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{3,t-2})\, u_t \rightarrow \sin(\tfrac{\pi}{2} s)\, \sigma\, [C_R \int_0^1 G_3(r)\, dB_{3s}(r) + C_I \int_0^1 G_4(r)\, dB_{3s}(r)]$$

$$- \cos(\tfrac{\pi}{2} s)\, \sigma\, [C_R \int_0^1 G_4(r)\, dB_{4s}(r) - C_I \int_0^1 G_3(r)\, dB_{4s}(r)] \qquad \text{(A.9.4)}$$

where C_R and C_I, respectively, denote the real and imaginary part of C(i).

Proof.

(i) When $z^1_{1t} = y_{1t} - y_{1,t-1}$ has a moving average representation as in (A.7), we can show that [see, e.g., Lee (1992, p. 34)]

$$y_{1,t} = \left[\sum_{i=0}^{\infty} C_i\right] \left[\sum_{j=1}^{t} u_i\right] + \left[\sum_{i=0}^{\infty} C_i \sum_{j=-i+1}^{0} u_j\right] - \left[\sum_{i=0}^{\infty} C_i \sum_{j=t-i+t}^{t} u_j\right] \qquad \text{(A.10.1)}$$

Using (A.1), it follows that:

$$N^{-2} \sum_{t=1}^{T} D_{st} y^2_{1,t-1} = C(1)^2 N^{-2} \sum_{t=1}^{T} D_{st} \left[\sum_{t=1}^{t-1} u_j\right]^2$$

$$+ o_p(1) = C(1)^2 N^{-2} \sum_{n=1}^{N} w_{1,n-1} + o_p(1).$$

The relation (A.8.1) now follows from (A.5) and the continuous mapping theorem. Using similar arguments, it can be shown that:

$$(-1)^t y_{2,t} = \left[\sum_{i=1}^{\infty} (-1)^i C_i\right] \left[\sum_{j=1}^{t} (-1)^j u_j\right] + \left[\sum_{i=0}^{\infty} (-1)^i C_i \sum_{j=-i+1}^{0} (-1)^j u_j\right]$$

$$- \left[\sum_{i=0}^{\infty} (-1)^i C_i \sum_{j=t-i+1}^{t} (-1)^j u_j\right]. \qquad \text{(A.10.2)}$$

From (A.1) for k = 2, we can show that:

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{2,t-1}^2 = C(-1)^2 N^{-2} \sum_{t=1}^{T} D_{st} \left[\sum_{j=1}^{t-1} (-1)^j u_j\right]^2 + o_p(1)$$

$$= C(-1)^2 N^{-2} \sum_{n=1}^{N} w_{2,n-1}^2 + o_p(1).$$

Using (A.5) for k = 2, the relation (A.8.2) can be obtained. When $z_{2t}^0 = (1 + B^2)\, y_{3,t}$ has a moving average representation $Z_{2t}^0 = C(B)u_t$, we can rewrite $y_{3,t}$ as [see Lee (1992, p. 34)]

$$y_{3,t} = C_R\, [C_t \sin(\tfrac{\pi}{2}t) - S_t\cos(\tfrac{\pi}{2}t)] - C_I[C_t\sin(\tfrac{\pi}{2}t) + S_t\sin(\tfrac{\pi}{2}t)] + o_p(T^{1/2}) \quad \text{(A.10.3)}$$

where $C_t = \sum_{j=1}^{t-1} \cos(\frac{\pi}{2} j)\, u_j$ and $S_t = \sum_{j=1}^{t-1} \sin(\frac{\pi}{2} j)\, u_j$.

Using (A.10.3), we can write:

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{3,t-1}^2 = N^{-2} \sum_{t=1}^{T} D_{st}\{C_R^2[C_t^2(\sin\tfrac{\pi}{2}\, t)^2 + S_t^2 \cos(\tfrac{\pi}{2}\, t)^2] + C_I^2[C_t^2 \cos(\tfrac{\pi}{2}\, t)^2$$

$$+ S_t^2 \sin(\tfrac{\pi}{2}\, t)^2] + C_R\, C_I[C_t\, S_t(\cos\tfrac{\pi}{2}\, t)^2 - C_t\, S_t(\sin\tfrac{\pi}{2}\, t)^2]\} + o_p(1)$$

$$= [\sin(\tfrac{\pi}{2}\, s)^2]\, N^{-2} \sum_{n=1}^{N} [C_R^2\, w_{4,n}^2 + C_I^2\, w_{3,n}^2 - C_R\, C_I\, w_{3,n}\, w_{4,n}]$$

$$+ [\cos(\tfrac{\pi}{2}\, s)]^2\, N^{-2} \sum_{n=1}^{N} [C_R^2\, w_{3,n}^2 + C_I^2\, w_{4,n}^2 + C_R\, C_I\, w_{3,n}\, w_{4,n}] + o_p(1).$$

Using (A.1) and (A.5) for k = 3.4, the relation (A.8.3) can be obtained for i = 1. A similar expression can be derived for $N^{-2} \sum_{t=1}^{T} D_{st} y_{3,t-2}^2$, which leads to the result in (A.8.4). Similarly, we can write:

$$N^{-2} \sum_{t=1}^{T} D_{st} y_{3,t-1} y_{3,t-2} = N^{-2} \sum_{t=1}^{T} D_{st}\{C_R^2[S_t C_{t-1}(\cos\tfrac{\pi}{2} t)^2 - C_t S_{t-1}(\sin\tfrac{\pi}{2} t)^2]$$

$$+ C_I^2[S_t C_{t-1} (\sin\tfrac{\pi}{2} t)^2 - C_t S_{t-1} (\cos\tfrac{\pi}{2} t)^2] + C_R C_I[C_t C_{t-1}(\sin\tfrac{\pi}{2} t)^2 - S_t S_{t-1}(\cos\tfrac{\pi}{2} t)^2]$$

$$+ C_R C_I [C_t C_{t-1}(\cos\tfrac{\pi}{2} t)^2 - S_t S_{t-1}(\sin\tfrac{\pi}{2} t)^2] + o_p(1).$$

When S is an even number, the above expression reduces to:

$$N^{-2} \sum_{t=1}^{T} D_{st}[C_R^2 S_t C_{t-1} - C_I^2 C_t S_{t-1} + C_R C_I(C_t C_{t-1} - S_t S_{t-1})].$$

Combining (A.1) and (A.5), the relation (A.8.4) follows from the continuous mapping theorem. The same argument applies to the case when S is an odd number. Note that while $N^{-2} \sum_{t=1}^{T} D_{st} y_{3,t-1} y_{3,t-2}$ converges to a nondegenerate asymptotic distribution in (A.8.4), the two series $y_{3,t-1}$ and $y_{3,t-2}$ are asymptotically uncorrelated in the sense that $\sum_{s=1}^{S} \left[N^{-2} \sum_{t=1}^{T} D_{st} y_{3,t-1} y_{3,t-2}\right] = op(1)$. This property is useful in deriving the asymptotic distribution (A.11.3) below.

(ii) Using (A.1) and (A.10), we obtain:

$$N^{-1} \sum_{t=1}^{T} D_{st} y_{1,t-1} u_t = C(1)\, N^{-1} \sum_{n=1}^{N} w_{1,n-1}\, e_{1,n}(s) + o_p(1)$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{2,t-1} u_t) = N^{-1} \sum_{t=1}^{T} D_{st}(-1)^{t-1} y_{2,t-1}(-1)^t u_t$$

$$= C(-1)\, N^{-1} \sum_{n=1}^{N} w_{2,n-1}\, e_{2,n}(s) + o_p(1)$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{3,t-1-i}\, u_t) = N^{-1} \sum_{n=1}^{N} D_{st}\{-C_R\, [C_{t-i} \sin\tfrac{\pi}{2}(t-i) - S_{t-i} \cos\tfrac{\pi}{2}(t-i)]$$

$$+ C_I\, [C_{t-i} \cos\tfrac{\pi}{2}(t-i) + S_{t-i} \sin\tfrac{\pi}{2}(t-i)]\}\, u_t + o_p(1).$$

Noting that $u_t = (-1)^{(t-1)/2} \cos(\frac{\pi}{2} t)\, u_t + (-1)^{(t-2)/2} \sin(\frac{\pi}{2} t)\, u_t$, we can write:

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y_{3,t-2}\, u_t) = N^{-1} \sum_{t=1}^{T} D_{st}\{(-C_R\, C_{t-1} + C_I\, S_{t-1})\, [\cos(\tfrac{\pi}{2} t)]^2\, u_t(-1)^{(t-1)/2}$$

$$+ (C_R\, S_{t-1} + C_I\, C_{t-1})\, [\sin(\tfrac{\pi}{2} t)]^2\, u_t(-1)^{(t-2)/2}\} + o_p(1).$$

A similar expression can be derived for $N^{-1} \sum_{t=1}^{T} D_{st}\, (-y_{3,t-1}\, u_t)$. Combining (A.1) and (A.5), the relations (A.9) can be obtained by using the continuous mapping theorem.

Proof of Theorem 2.1.

(i) Using standard arguments, it can be shown that:

$$W^R(\phi) = \sum_{s=1}^{S} (\hat{\alpha}_s - \phi)^2 / \hat{\sigma}^2 \left[\sum_{t=1}^{T} D_{st}\, X_t\, X_t'\right]^{11}$$

$$= \sum_{s=1}^{S} \left[\sum_{t=1}^{T} D_{st}\, y_{t-1}\, u_t\right]^2 / \sigma^2 \left[\sum_{t=1}^{T} D_{st}\, y_{t-1}^2\right] + o_p(1).$$

For the zero frequency case where the null hypothesis is that $H_0^R(1)$: $\alpha_s = 1$ for all $s = 1, \ldots, S$, the relations (A.8.1) and (A.9.1) can be used to derive

$$\psi_S^R(1) = \sum_{s=1}^{S} [\textstyle\int_0^1 G_1(r)\, d\, B_{1s}(r)]^2 / [\textstyle\int_0^1 G_1(r)^2\, dr]. \tag{A.11.1}$$

Similarly, for testing $H_0^R(-1)$: $\alpha_s = -1$ for all s, the relations (A.8.2) and (A.9.2) can be used to show:

$$\psi_S^R(-1) = \sum_{s=1}^{S} [\int_0^1 G_2(r)\, d\, B_{2s}(r)]^2 / [\int_0^1 G_2(r)^2\, dr]. \qquad \text{(A.11.2)}$$

Noting that $G_1(r)$ and $G_2(r)$ are independent and $B_{2s}(r) = (-1)^s B_{1s}(r)$, it follows that $\psi_S^R(1)$ and $\psi_S^R(-1)$ have the same distribution denoted ψ_S^R. Thus, we can use the same critical values when we are interested in testing for real unit roots, either − 1 or 1.

(ii) To prove $\psi_S^C(s)$ for complex unit roots, we first consider the test statistics under $H_0^C(0)$: $\gamma_{1s} = 0$, $\gamma_{2s} = 1$ for all s = 1, ..., S. In this case, the Wald statistic can be written as:

$$W_S^C(0) = \sum_{s=1}^{S} (\hat{\gamma}_{1s}, \hat{\gamma}_{2j} - 1) \left\{ \left[\sum_{t=1}^{T} D_{st} X_{3t} X_{3t}' \right]^{-1} \right\}_{1:2,1:2} (\hat{\gamma}_{1s}, \hat{\gamma}_{2s} - 1)' / \hat{\sigma}^2$$

$$= \sum_{s=1}^{S} \left[\sum_{t=1}^{T} D_{st} (y_{t-1} u_t, y_{t-2} u_t) \right] \left\{ \left[\sum_{t=1}^{T} D_{st} X_{3t} X_{3t}' \right]^{-1} \right\}_{1:2,1:2}$$

$$\left[\sum_{t=1}^{T} D_{st} (y_{t-1} u_t, y_{t-2} u_t) \right]' / \sigma^2 + o_p(1)$$

where $X_{3t} = (-y_{t-1} - y_{t-2}, z_{2,t-1}^0, \ldots, z_{2,t-p+2}^0)'$.

$$= \mathrm{tr}\left\{\begin{bmatrix}\Sigma D_{1t}(y_{t-1} u_t, y_{t-2} u_t)\\ \Sigma D_{2t}(y_{t-1} u_t, y_{t-2} u_t)\\ \vdots \\ \Sigma D_{St}(y_{t-1} u_t, y_{t-2} u_t)\end{bmatrix} \left[\sum_{t=1}^{T} D_t \otimes (y_{t-1}\ y_{t-2})'(y_{t-1}\ y_{t-2})\right]^{-1} \times \right.$$

$$\left.\begin{bmatrix}\Sigma D_{1t}(y_{t-1} u_t, y_{t-2} u_t)\\ \Sigma D_{2t}(y_{t-1} u_t, y_{t-2} u_t)\\ \vdots \\ \Sigma D_{St}(y_{t-1} u_t, y_{t-2} u_t)\end{bmatrix}'\right\} + o_p(1)$$

where $D_t = \mathrm{diag}(D_{1t}, D_{2t}, ..., D_{St})$.

Using the relations (A.8.3) - (A.8.4) and (A.9.3) - (A.9.4), we can show that:

$$W_S^C(0) \to \mathrm{tr}\{\textstyle\int_0^1 dW\, (G_3, G_4)\, [\int_0^1 (G_3, G_4)]^{-1} \int_0^1 (G_3, G_4)'\, dW'\} \qquad \text{(A.11.3)}$$

where W(r) is as defined in (A.4.1). The derivation is tedious as the limiting distributions in (A.8.3) - (A.8.4) and (A.9.3) - (A.9.4) depend on the value of S. In the simple case, when $C_R = 1$ and $C_I = 0$, i.e., $(1 + B^2)\, y_t = u_t$; however, the relation (A.11.3) can be obtained by straightforward application of the results in Lemma A.1 and the continuous mapping theorem.

Noting that $G_3(r) = \sum_{j=1}^{S/2} (-1)^{j-1} W_{2j-1}(r)$ and $G_4(r) = \sum_{j=1}^{S/2} (-1)^j W_{2j}(r)$, it is convenient to rewrite (A.11.3) as:

$$W_S^C(0) \to \psi_S^C(0)\, \mathrm{tr}\{\textstyle\int_0^1 (dG)\, G_{34}'\, [\int_0^1 G_{34}, G_{34})]^{-1} \int_0^1 G_{34}\, (dG)'\} \qquad \text{(A.12)}$$

where G(r) is an S-dimensional standard Brownian motion, the first four elements of which are $G_{1234} = \left[(\sqrt{S})^{-1} G_1, (\sqrt{S})^{-1} G_2, \sqrt{2}(\sqrt{S})^{-1} G_3, \sqrt{2}(\sqrt{S})^{-1} G_4\right]'$, and $G_{34}(r)$ is a 2×1 vector with the third and fourth elements of G(r).[3]

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[3] G(r) can be obtained from W(r) by multiplying an orthogonal matrix. Its first four columns are: $(\sqrt{S})^{-1}(1, 1, ..., 1)'$, $(\sqrt{S})^{-1}(-1, 1, ..., 1)'$, $\sqrt{2}(\sqrt{S})^{-1}(1, 0, -1, ..., 0)'$ and $\sqrt{2}(\sqrt{S})^{-1}(0, -1, 0, ..., 1)'$.

Next, we show that the limiting distributions for testing the unit roots associated with the polynomial $(1 + \phi B + B^2)$ do not depend on the value of $\phi = 2\cos\theta$. When the hypothesis of interest in the regression model (2.3) is given by $H_0^C(\phi)$: $\gamma_{1s} = \phi$, $\gamma_{2s} = 1$ for all s = 1, ..., S, it can be shown that testing $H_0^C(\phi)$ in (2.3) is equivalent to testing whether $\gamma^*_{1s} = 0$ and $\gamma^*_{2s} = 1$ hold in the regression:

$$y_t = \gamma^*_{1t}(-y^*_{t-1}) + \gamma^*_{2t}(-y^*_{t-2}) + \sum_{j=1}^{p-2} z^{\phi}_{2,t-j} + u_t$$

where $y^*_{t-1} = \sin\theta\, y_{t-1}$. $y^*_{t-2} = y_{t-2} - 2\cos\theta\, y_{t-1}$ and $z^{\phi}_{2,t} = (1 + \phi B + B^2)\, y_t$. Notice first that when $\theta = \frac{\pi}{2}$ the above regression model reduces to (2.3) and, hence, that the hypothesis H_0^{C*}: $\gamma^*_{1s} = 0$, $\gamma^*_{2s} = 1$ reduces in this case to $H_0^C(0)$: $\gamma_{1s} = 0$, $\gamma_{2s} = 1$. In general, the hypothesis H_0^{C*} in the above regression can be shown to be equivalent to $\gamma_{2s} = \gamma^*_{2s} = 1$ and $\gamma_{1s} = \gamma^*_{1s} \sin\theta + \gamma^*_{2s}(2\cos\theta) = 2\cos\theta = \phi$.

Now, consider:

$$W_S^C(\phi) = \sum_{s=1}^{S} (\hat{\gamma}_{1s}, \hat{\gamma}_{2s} - 1) \left\{ \left[\sum_{t=1}^{T} D_{st} X_{\phi t} X'_{\phi t} \right]^{-1} \right\}_{1:2,1:2} (\hat{\gamma}_{1s} - \phi, \hat{\gamma}_{2s} - 1)' / \hat{\sigma}^2$$

$$= \sum_{s=1}^{S} (\hat{\gamma}^*_{1s}, \hat{\gamma}^*_{2s} - 1) \{[\Sigma D_{st} X^*_{\phi t} X^{*\prime}_{\phi t}]^{-1}\}_{1:2,1:2} (\hat{\gamma}^*_{1s}, \hat{\gamma}^*_{2s} - 1)' / \hat{\sigma}^2$$

$$= \sum_{s=1}^{S} \left\{ \left[\sum_{t=1}^{T} D_{st} (y^*_{t-1}, u_t, y^*_{t-2} u_t) \right] \left[\sum_{t=1}^{T} D_{st} (y^*_{t-1}, y^*_{t-2})' (y^*_{t-1}, y^*_{t-2}) \right]^{-1} \times \right.$$

$$\left[\sum_{t=1}^{T} D_{st} (y^*_{t-1} u_t, y^*_{t-2} u_t) \right] / \sigma^2 + o_p(1)$$

where $X_{\phi t} = (-y_{t-1}, -y_{t-2}, z^{\phi}_{2,t-1}\ z^{\phi}_{2,t-p+2})'$ and $X^*_{\phi t} = (-y^*_{t-1}, -y^*_{t-2}, z^{\phi}_{2,t-1}, ..., z^{\phi}_{2,t-p+2})'$.

Using the relations:

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y^*_{t-1}\, u_t) = \sin\theta\, N^{-1} \sum_{t=1}^{T} D_{st}(-y_{t-1}\, u_t)$$

$$N^{-1} \sum_{t=1}^{T} D_{st}(-y^*_{t-2}\, u_t) = N^{-1} \sum_{t=1}^{T} D_{st}(-y_{t-2}\, u_t) - 2\cos\theta\, N^{-1} \sum_{t=1}^{T} D_{st}(-y_{t-1}\, u_t)$$

$$N^{-2} \sum_{t=1}^{T} D_{st} -y^{*2}_{t-1} = \sin^2\theta\, N^{-2} \sum_{t=1}^{T} D_{st}\, y^2_{t-1}$$

$$N^{-1} \sum_{t=1}^{T} D_{st}\, y^{*2}_{t-2} = N^{-2} \sum_{t=1}^{T} D_{st}\, y^2_{t-2} + 4\cos^2\theta\, N^{-2} \sum_{t=1}^{T} D_{st}\, y^2_{t-1}$$

$$- 4\cos\theta\, N^{-2} \sum_{t=1}^{T} D_{st}\, y_{t-1}\, y_{t-2},$$

it can be shown that:

$$\left[\sum_{t=1}^{T} D_{st}(y^*_{t-1}\, u_t,\, y^*_{t-2}\, u_t)\right] \left[\sum_{t=1}^{T} D_{st}(y^*_{t-1},\, y^*_{t-2})'\,(y^*_{t-1},\, y^*_{t-2})\right]^{-1} \left[\sum_{t=1}^{T} D_{st}(y^*_{t-1}\, u_t,\, y^*_{t-2}\ _t)'\right]$$

$$= \left[\sum_{t=1}^{T} D_{st}(y_{t-1}\, u_t,\, y_{t-2}\, u_t)\right] \left[\sum_{t=1}^{T} D_{st}(y_{t-1},\, y_{t-2})'\,(y_{t-1},\, y_{t-2})\right]^{-1} \left[\sum_{t=1}^{T} D_{st}(y_{t-1}\, u_t,\, y_{t-2}\, u_t)'\right].$$

Therefore, we have:

$$W^C_S(\phi) \to \psi^C_S = \mathrm{tr}\{\textstyle\int_0^1 dG\, G'_{34}[\int_0^1 G_{34}\, G'_{34}]^{-1} \int_0^1 G_{34}\, dG'\},$$

which is independent of the value of ϕ.

To prove Theorem 2.2, we will use the results in the following lemma.

Lemma A.2. As $T \to \infty$ (and hence $N \to \infty$), we have:

(i) $$N^{-1/2} \sum_{t=1}^{T} D_{st} u_t \rightarrow \sigma B_{1s}(1) \qquad \text{(A.13.1)}$$

$$N^{-1/2} \sum_{t=1}^{T} D_{st}(-1)^t u_t \rightarrow \sigma B_{2s}(1) \qquad \text{(A.13.2)}$$

$$N^{-1/2} \sum_{t=1}^{T} D_{st} \sin(\tfrac{\pi}{2}t) u_t \rightarrow \sigma B_{3s}(1) \qquad \text{(A.13.3)}$$

$$N^{-1/2} \sum_{t=1}^{T} D_{st} \cos(\tfrac{\pi}{2}t) u_t \rightarrow \sigma B_{4s}(1) \qquad \text{(A.13.4)}$$

(ii) $$N^{-3/2} \sum_{t=1}^{T} D_{st}\, y_{1,t-1} \rightarrow C(1)\, \sigma \int_0^1 G_1(r)\, dr \qquad \text{(A.14.1)}$$

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{2,t-1}) \rightarrow (-1)^s C(-1)\, \sigma \int_0^1 G_2(r)\, dr \qquad \text{(A.14.2)}$$

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{3,t-1}) \rightarrow \cos(\tfrac{\pi}{2}s)\, \sigma [C_R \int_0^1 G_3(r)\, dr + C_I \int_0^1 G_4(r)\, dr$$

$$- \sin(\tfrac{\pi}{2}s)\, \sigma [C_R \int_0^1 G_4(r)\, dr - C_I \int_0^1 G_3(r)\, dr] \qquad \text{(A.14.3)}$$

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{3,t-2}) \rightarrow \sin(\tfrac{\pi}{2}s)\, \sigma [C_R \int_0^1 G_3(r)\, dr + C_I \int_0^1 G_4(r)\, dr$$

$$- \cos(\tfrac{\pi}{2}s)\, \sigma [C_R \int_0^1 G_4(r)\, dr - C_I \int_0^1 G_3(r)\, dr] \qquad \text{(A.14.4)}$$

(iii) $$N^{-5/2} \sum_{t=1}^{T} D_{st}\, n\, y_{1,t-1} \rightarrow C(1)\, \sigma \int_0^1 r\, G_1(r)\, dr \qquad \text{(A.15.1)}$$

$$N^{-5/2} \sum_{t=1}^{T} D_{st}\, n\, (-y_{t-1}) \rightarrow (-1)^s C(-1)\, \sigma \int_0^1 r\, G_2(r)\, dr \qquad \text{(A.15.2)}$$

(iv) $$N^{-3/2} \sum_{t=1}^{T} D_{st}\, n\, u_t \rightarrow \sigma[B_{1s}(1) - \int_0^1 B_{1s}(r)\, dr] \qquad \text{(A.16.1)}$$

$$N^{-3/2} \sum_{t=1}^{T} D_{st}\, (-1)^t u_t \rightarrow \sigma[B_{2s}(1) - \int_0^1 B_{2s}(r)\, dr] \qquad \text{(A.16.2)}$$

Proof.

(i) The relations in (A.13) follow immediately from (A.1) and (A.4).

(ii) Using (A.1) and (A.10), it can be shown that:

$$N^{-3/2} \sum_{t=1}^{T} D_{st}\, y_{1,t-1} = C(1)\, N^{-3/2} \sum_{t=1}^{T} D_{st} \left[\sum_{j=1}^{t-1} u_j\right] + o_p(1)$$

$$= C(1)\, N^{-3/2} \sum_{n=1}^{N} w_{1,n-1} + o_p(1).$$

To show (A.14.2), define $w_{2,n-1}(s) = (-1)^s \sum_{j=1}^{(n-1)S+s} (-1)^j u_j$. Then, we have:

$$w_{2,n-1}(s) = (-1)^s \left[w_{2,n-1} + \sum_{j=1}^{S} (-1)^j u_{(n-1)S+j}\right].$$

Thus, we obtain:

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{2,t-1}) = C(-1)\, N^{-3/2} \sum_{t=1}^{T} D_{st}\, (-1) \left[\sum_{j=1}^{t-1} (-1)^j u_j\right] + o_p(1)$$

$$= C(-1)\, N^{-3/2} \sum_{n=1}^{N} (-1)\, w_{2,n-1(s-1)} + o_p(1)$$

$$= C(-1)\, N^{-3/2} \sum_{n=1}^{N} (-1)^s\, w_{2,n-1} + o_p(1).$$

The relations (A.14.3) and (A.14.4) can similarly be obtained from (A.10.3) and the continuous mapping theorem. That is,

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{3,t-1}) \begin{cases} = (-1)^{(s+1)/2} N^{-3/2} \sum_{n=1}^{N} (C_R w_{4,n} - C_I w_{3,n}) + o_p(1), \text{ for s odd} \\ = (-1)^{s/2} N^{-3/2} \sum_{n=1}^{N} (C_R w_{3,n} - C_I w_{4,n}) + o_p(1), \text{ for s even} \end{cases}$$

$$N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{3,t-2}) \begin{cases} =(-1)^{(s-1)/2} N^{-3/2} \sum_{n=1}^{N} [C_R w_{3,n} - C_I w_{4,n}) + o_p(1), \text{ for s odd} \\ = (-1)^{(s/2)/2} N^{-3/2} \sum_{n=1}^{N} (C_R w_{4,n} - C_I w_{3,n}) + o_p(1), \text{ for s even.} \end{cases}$$

(iii) Using similar arguments, one can show that:

$$N^{-5/2} \sum_{t=1}^{T} D_{st}\, n\, y_{1,t-1} = C(1)\, N^{-3/2} \sum_{n=1}^{N} \left[\frac{n}{N}\right] w_{1,n-1} + o_p(1)$$

$$N^{-5/2} \sum_{t=1}^{T} D_{st}\, n\, (-y_{2,t-1}) = (-1)^s\, C(-1)\, N^{-3/2} \sum_{n=1}^{N} \left[\frac{n}{N}\right] w_{2,n-1} + o_p(1).$$

The relations in (A.15) follow from (A.2), (A.4) and Phillips and Perron (1988). Similar expressions can be obtained for $y_{3,t}$, which are suppressed here, as they are not explicitly used in the proof of Theorem 2.2.

(iv) Similarly, one can also show that

$$N^{-3/2} \sum_{t=1}^{T} D_{st}\, n\, u_t = N^{-1/2} \sum_{n=1}^{N} \left[\frac{n}{N}\right] e_{1n(s)} \rightarrow \sigma[B_{1s}(1) - \int_0^1 B_{1s}(r)\, dr].$$

The other result given in the lemma follows by similar arguments.

Proof of Theorem 2.2.

(i) We follow the steps of the proof of Theorem 2.1. It can be shown that for $\phi = 1$:

$$W^R_{S\mu}(\phi) = \sum_{s=1}^{S} \left[\sum_{t=1}^{T} D_{st}(y_{t-1} - \bar{y}_s)(u_t - \bar{u}_s)\right]^2 / \sigma^2 \left[\sum_{t=1}^{T} D_{st}(y_{t-1} - \bar{y}_s)^2\right] + o_p(1)$$

(A.17)

where $\bar{y}_s = N^{-1} \sum_{t=1}^{T} D_{st}\, y_{t-1}$, $\bar{u}_s = N^{-1} \sum_{t=1}^{T} D_{st}\, u_t$.

Using standard arguments, it is easy to rewrite (A.17) as:

$$W^R_{S\mu}(\phi) = \sum_{s=1}^{S} \left\{\left[N^{-1} \sum_{t=1}^{T} D_{st}\, y_{t-1}\, u_t - \left[N^{-3/2} \sum_{t=1}^{T} D_{st}\, y_{t-1}\right] \left[N^{-1/2} \sum_{t=1}^{T} D_{st}\, u_t\right]\right]\right.$$

$$\left.\left[N^{-2} \sum_{t=1}^{T} D_{st}\, y^2_{t-1} - \left[N^{-3/2} \sum_{t=1}^{T} D_{st}\, y_{t-1}\right]^2\right]^{-1}\right\} + o_p(1) \qquad \text{(A.18)}$$

Combining the results in Lemmas A.1 and A.2, i.e., (A.8.1), (A.9.1), (A.13.1), (A.14.1), we obtain the formula for $\psi^R_{S\mu}(\phi)$:

$$\psi^R_{S\mu}(\phi) = \sum_{s=1}^{S} [\textstyle\int_0^1 G_1(r)\, d\, B_{1s}(r) - \int_0^1 G_1(r)\, dr\, B_{1s}(1)]^2 / [\int_0^1 G_1(r)^2\, dr - (\int_0^1 G_1(r)\, dr)^2].$$

(A.19.1)

As for $W^R_{S\mu}(\phi)$ with $\phi = -1$, we can use similar arguments to obtain:

$$W^R_{S\mu}(-1) = \sum_{s=1}^{S} \left\{N^{-1} \sum_{t=1}^{T} D_{st}(-y_{t-1}\, u_t) - \left[N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{t-1})\right] \left[N^{-1/2} \sum_{t=1}^{T} D_{st}\, u_t\right]\right\}$$

$$\sigma^{-2}\left\{N^{-2} \sum_{t=1}^{T} D_{st}\, y^2_{t-1} - \left[N^{-3/2} \sum_{t=1}^{T} D_{st}(-y_{t-1})\right]^2\right\}^{-1} + o_p(1).$$

Noting that $N^{-1/2} \sum_{t=1}^{T} D_{st}\, u_t = (-1)^s\, N^{-1/2} \sum_{t=1}^{T} D_{st}(-1)\, u_t$, the formula for $\psi^R_{S\mu}$ can then be obtained as:

$$\psi^R_{S\mu}(-1) = \sum_{s=1}^{S} [\textstyle\int_0^1 G_2(r)\, d\, B_{2s}(r) - \int_0^1 G_2(r)\, dr\, B_{2s}(1)]^2 / [\int_0^1 G_2(r)^2\, dr - (\int_0^1 G_2(r)\, dr)^2]$$

(A.19.2)

As for $W^C_{S\mu}(\phi)$, we use arguments similar to (A.18) where y_{t-1}, y_{t-2} and u_t in the Wald statistic need to be replaced by their "demeaned" counterparts, i.e., $(y_{t-1} - \bar{y}_{-1,s})$, $(y_{t-2} - \bar{y}_{-2,s})$ and $(u_t - \bar{u}_s)$, respectively. The results in Lemma A.1 and A.2 can be used to obtain:

$$W^C_{S\mu} \to \psi^C_{S\mu} = \text{tr}\,\{\textstyle\int_0^1 dG\, F'_{34}[\int_0^1 F_{34}\, F'_{34}]^{-1} \int_0^1 F_{34}\, dG'\} \tag{A.19.3}$$

where $F_{34}(r) = G_{34}(r) - \int_0^1 G_{34}(r)\,dr$ is a 2-dimensional Brownian motion, which is the demeaned counterpart of $G_{34}(r)$.

(ii) As for the test statistics for the regression models with an intercept and a linear time trend, we can show, for instance, that [see, e.g., Phillips and Perron (1998)]

$$W^R_{S\tau}(\phi) = \sum_{s=1}^{S} M_s(\hat{\alpha}_s - 1)^2 / \hat{\sigma}^2[N^2(N^2 - 1)/12] + o_p(1)$$

where

$$\hat{\alpha}_s - 1 = M_s^{-1}\Bigg\{[N(N+1)/2] \sum_{t=1}^{T} D_{st}\, n\, y_{t-1} \sum_{t=1}^{T} D_{st}\, u_t$$

$$- [N(N+1)(2N+1)/6] \sum_{t=1}^{T} D_{st}\, y_{t-1} \sum_{t=1}^{T} D_{st}\, u_t$$

$$- N \sum_{t=1}^{T} D_{st}\, n\, y_{t-1} \sum_{t=1}^{T} D_{st}\, n\, u_t + [N(N+1)/2] \sum_{t=1}^{T} D_{st}\, y_{t-1} \sum_{t=1}^{T} D_{st}\, n\, u_t$$

$$+ [N^2(N^2-1)/12] \sum_{t=1}^{T} D_{st}\, y_{t-1}\, u_t\Bigg\} + o_p(1)$$

$$M_s = [N^2(N^2-1)/12] \sum_{t=1}^{T} D_{st}\, y^2_{t-1} - N\left[\sum_{t=1}^{T} D_{st}\, n\, y_{t-1}\right]^2$$

$$+ N(N+1) \sum_{t=1}^{T} D_{st}\, n\, y_{t-1} \sum_{t=1}^{T} D_{st}\, y_{t-1} - [N(N+1)(2N+1)/6] \left[\sum_{t=1}^{T} D_{st}\, n\, y_{t-1}\right]^2$$

Combining the results for $y_{1,t-1}$ in Lemmas A.1 and A.2, we obtain:

$$W^R_{S\tau}(\phi) \rightarrow \psi^R_{S\tau} = \sum_{s=1}^{S} A_s^2 / D \qquad \text{(A.20.1)}$$

where

$$A_s = 6\, B_{1s}(1) \int_0^1 r\, G_1(r)\, dr - 4\, B_{1s}(1) \int_0^1 G_1(r)\, dr$$

$$- 12[B_{1s}(1) - \int_0^1 B_{1s}(r)\, dr]\, [\int_0^1 r\, G_1(r)\, dr - \tfrac{1}{2} \int_0^1 G_1(r)\, dr] + \int_0^1 G_1(r)\, dB_{1s}(r),$$

$$D = \int_0^1 G_1(r)^2\, dr - 12\, [\int_0^1 r\, G_1(r)\, dr]^2 + 12 \int_0^1 G_1(r)\, dr \int_0^1 r\, G_1(r)\, dr$$

$$- 4\, [\int_0^1 G_1(r)\, dr]^2$$

The same formula can be obtained for $W^R_{S\tau}(-1)$ except that $G_1(r)$ and $B_{1s}(r)$ should be replaced by $G_2(r)$ and $B_{2s}(r)$. As mentioned in the proof of Theorem 2.1, $W^R_{s\tau}(1)$ and $W^R_{S\tau}(-1)$ have the same distribution, and the same critical values can be used to test for real unit roots ±1. As for the limiting distribution of $W^C_{S\tau}(\phi)$, it should be noted first that A_s and D in (A.20.1) can be rewritten as:

$$A_s = \int_0^1 G_1^*(r)\, dB_{1s}(r) \text{ and } D = \int_0^1 G_1^*(r)^2\, dr$$

where $G_1^*(r)$ is a "detrended" Brownian motion such that

$$G_1^*(r) = G_1(r) - 4\, [\int_0^1 G_1(t)\, dt - \tfrac{3}{2} \int_0^1 t\, G_1(t)\, dt + 6r[\int_0^1 G_1(t)\, dt - 2\int_0^1 t\, G_1(t)\, dt].$$

Using similar arguments to the derivation of $\psi^C_{S\mu}$ in (A.19.3), it can be shown that:

$$\psi^C_{S\tau} = \mathrm{tr}\{\int_0^1 dG\, H'_{34}\, [\int_0^1 H_{34}\, H'_{34}]^{-1} \int_0^1 H_{34}\, dG'\} \qquad \text{(A.20.2)}$$

where

$$H_{34}(r) = G_{34}(r) - 4\,[\int_0^1 G_{34}(t)\,dt - \tfrac{3}{2}\int_0^1 t\,G_{34}(t)\,dt] + 6r[\int_0^1 G_{34}(t)\,dt - 2\int_0^1 t\,G_{34}(t)\,dt].$$

Using analogous arguments to the proof of Theorem 2.1, it follows that for any arbitrary choice of ϕ such that $|\phi| < 2$:

$$W^C_{S\mu}(\phi) \to \psi^C_{S\mu}$$

$$W^C_{S\tau}(\phi) \to \psi^C_{S\tau}$$

Proof of Theorem 3.1.

As in the proof of Theorem 2.1, one can show that:

$$W^A_4 = \sum_{s=1}^{S} (\hat{\pi}_{1s}, \hat{\pi}_{2s}, \hat{\pi}_{3s}, \hat{\pi}_{4s}) \left\{\left[\sum_{t=1}^{T} D_{st} X_{4t} X'_{4t}\right]^{-1}\right\}_{1:4,1:4} (\hat{\pi}_{1s}, \hat{\pi}_{2s}, \hat{\pi}_{3s}, \hat{\pi}_{4s})' / \hat{\sigma}^2$$

$$= \sum_{s=1}^{S} \left[\sum_{t=1}^{T} D_{st}(y_{1,t-1}\, u_t, y_{2,t-1}\, u_t, y_{3,t-1}\, u_t, y_{3,t-2}\, u_t)\right] \left\{\left[\sum_{t=1}^{T} D_{st} X_{4t} X'_{4t}\right]^{-1}\right\}_{1:4,1:4}$$

$$\left[\sum_{t=1}^{T} D_{st}\,(y_{1,t-1}\, u_t, y_{2,t-1}\, u_t, y_{3,t-1}\, u_t, y_{3,t-2}\, u_t\right]' + o_p(1).$$

Moreover, according to the proof of Lemma A.1, it follows that:

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{1,t-1}(-y_{2,t-1}) \to (-1)^S\, C(1)\, C(-1)\, \sigma^2 \int_0^1 G_1(r)\, G_2(r)\, dr$$

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{1,t-1}(-y_{3,t-1-i}) \to \cos[\tfrac{\pi}{2}(s-i)]\, C(1)\, \sigma^2 \times$$

$$[\int_0^1 G_1(r)\, G_3(r)\, dr + C_I \int_0^1 G_1(r)\, G_4(r)\, dr]$$

$$- \sin[\tfrac{\pi}{2}(s+i)]\, C(1)\, \sigma^2\, [C_R \int_0^1 G_1(r)\, G_4(r)\, dr - C_I \int_0^1 G_1(r)\, G_3(r)\, dr]$$

$$N^{-2} \sum_{t=1}^{T} D_{st}\, y_{2,t-1}\, y_{3,t-1-i} \to (-1)^s \cos[\tfrac{\pi}{2}(s-i)]\, C(-1)\, \sigma^2 \times$$

$$[C_R \int_0^1 G_2(r)\, G_3(r)\, dr + C_I \int_0^1 G_2(r)\, G_4(r)\, dr]$$

$$- (-1)^s \sin[\tfrac{\pi}{2}(s-i)]\, C(-1)\, \sigma^2\, [C_R \int_0^1 G_2(r)\, G_4(r)\, dr - C_I \int_0^1 G_2(r)\, G_3(r)\, dr].$$

Note that while $N^{-2} \sum_{t=1}^{T} D_{st}\, y_{k,t},\, y_{j,t}$ $(k, j = 1, 2, 3, k \neq j)$ have a nondegenerating asymptotic distribution for each season, they are uncorrelated asymptotically so that:

$$T^{-2} \sum_{t=1}^{T} y_{k,t}\, y_{j,t} = \sum_{s=1}^{S} \left[N^{-2} \sum_{t=1}^{T} D_{st}\, y_{k,t}\, y_{j,t} \right] = o_p(1).$$

Using this property, it can be shown that:

$$W_4^A = \mathrm{tr}\left\{ \begin{bmatrix} \sum_{t=1}^{T} D_{1t}(y_{1,t-1}\, u_t,\, y_{2,t-1}\, u_t,\, y_{3,t-1}\, u_t,\, y_{3,t-2}\, u_t) \\ \vdots \\ \sum_{t=1}^{T} D_{4t}(y_{1,t-1}\, u_t,\, y_{2,t-1}\, u_t,\, y_{3,t-1}\, u_t,\, y_{3,t-2}\, u_t) \end{bmatrix} \times \right.$$

$$\left[\sum_{t=1}^{T} D_{4t}(y_{1,t-1},\, y_{2,t-1},\, y_{3,t-1},\, y_{4,t-1})'\, (y_{1,t-1},\, y_{2,t-1},\, y_{3,t-1},\, y_{3,t-2}) \right]^{-1} \times$$

$$\left. \begin{bmatrix} \sum_{t=1}^{T} D_{1t}(y_{1,t-1}\, u_t,\, \ldots,\, y_{3,t-1}\, u_t) \\ \vdots \\ \sum_{t=1}^{T} D_{4t}(y_{1,t-1}\, u_t,\, \ldots,\, y_{3,t-2}\, u_t) \end{bmatrix}' \right\} / \sigma^2 + o_p(1).$$

The relations in Lemmas A.1 and A.2 can then be used to obtain:

$$W_4^A \to \mathrm{tr}\{ \int_0^1 dW\, G'\, [\int_0^1 G\, G']^{-1} \int_0^1 G\, dW' \}$$

where G(r) is a 4-dimensional standard Brownian motion such that $G = (1/2\ G_1, 1/2\ G_2, 1/\sqrt{2}\ G_3, 1/\sqrt{2}\ G_4)$. Then as in (A.12), the above expression can be rewritten as:

$$W_4^A \to \psi_4^A = tr\{\int_0^1 (dG)\ G'\ [\int_0^1 G\ G']^{-1} \int_0^1 G\ (dG)'\}.$$

Note that the Wald statistic W_4^A for the hypothesis that $\pi_{1s} = \pi_{2s} = \pi_{3s} = \pi_{4s} = 0$ for all $s = 1, ..., 4$ has the same asymptotic distribution as the Johanson's test statistic for cointegration with $(n - r) = 4$. See Table 1 in Johanson (1988, p. 239).

As for the Wald statistic W_4^S, we can first show that:

$$W_4^S = \sum_{j=1}^{S} (\hat{\pi}_{2s}\ \hat{\pi}_{3s}\ \hat{\pi}_{4s}) \left\{\left[\sum_{t=1}^{T} D_{st}\ X_{4t}\ X'_{4t}\right]^{-1}\right\}_{2:4,2;4} (\hat{\pi}_{2s}\ \hat{\pi}_{3s}\ \hat{\pi}_{4s})' / \hat{\sigma}^2$$

$$= \sum_{s=1}^{S} \left[\sum_{t=1}^{T} D_{st}(y_{2,t-1}\ u_t, y_{3,t-1}\ u_t, y_{3,t-2}\ u_t)\right] \left\{\left[\sum_{t=1}^{T} D_{st}\ X_{4t}\ X'_{4t}\right]^{-1}\right\}_{2:4,2:4}$$

$$\left[\sum_{t=1}^{T} D_{st}(y_{2,t-1}\ u_t, y_{3,t-1}\ u_t, y_{3,t-2}\ u_t)\right]' / \sigma^2 + o_p(1).$$

Then, it can be shown that:

$$W_4^S \to tr\{\int_0^1 (dW)\ G'_{234}\ [\int_0^1 G_{234}\ G'_{234}]^{-1} \int_0^1 G_{234}(dW)'\}$$

where $G_{234}(r) = (1/2\ G_2(r), 1/\sqrt{2}\ G_3(r), 1/\sqrt{2}\ G_4(r))'$. By multiplying an orthogonal matrix to W (see footnote 1 for details), we can show that:

$$W_4^S \to \psi_4^S = tr\{\int_0^1 (dG)\ G'_{234}\ [\int_0^1 G_{234}\ G'_{234}]^{-1} \int_0^1 G_{234}(dG)'\}.$$

As for the test statistics for the regression models which contain deterministic terms (intercept and time trend), we can use developments similar to the proof of Theorem 2.2 except that the Brownian motion process G(r) needs to be replaced by its "demeaned" and "detrended" versions, respectively. That is,

$$W^A_{4\mu} \to \psi^A_{4\mu} = \mathrm{tr}\{\textstyle\int_0^1 (dG)\, F' \,[\int_0^1 F\, F']^{-1} \int_0^1 F(dG)'\}$$

$$W^S_{4\tau} \to \psi^S_{4\tau} = \mathrm{tr}\{\textstyle\int_0^1 (dG)\, H' \,[\int_0^1 H\, H']^{-1} \int_0^1 H(dG)'\}$$

where

$$F(r) = G(r) - \textstyle\int_0^1 G(r)\, dr$$

and

$$H(r) = F(r) - 12(r - \tfrac{1}{2}) \textstyle\int_0^1 (t - \tfrac{1}{2})\, F(t)\, dt.$$

It should be noted that $W^A_{4\mu}$ has the same asymptotic distribution as the LR statistic for cointegration in Johanson and Juselius (1990, Table A.2), and that $W^A_{4\tau}$ has the same distribution as $TR_\tau(n - r)$ statistic in Perron and Campbell (1993, p. 787) with $(n - r) = 4$.

Similar expressions can be obtained for $W^S_{4\mu}$ and W^S_{4t}, namely:

$$W^S_{4\mu} \to \psi^S_{4\mu} = \mathrm{tr}\{\textstyle\int_0^1 (dG)\, F'_{234} \,[\int_0^1 F_{234}\, F'_{234}]^{-1} \int_0^1 F_{234}(dG)'\}$$

$$W^S_{4\tau} \to \psi^S_{4\tau} = \mathrm{tr}\{\textstyle\int_0^1 (dG)\, H'_{234} \,[\int_0^1 H_{234}\, H'_{234}]^{-1} \int_0^1 H_{234}(dG)'\}.$$

To conclude, we turn our attention to the monthly regression models. To do so, first, we define an appropriate set of filtered series:

$$y_{1t} = (1 + B + B^2 + B^3 + B^4 + B^5 + B^6 + B^7 + B^8 + B^9 + B^{10} + B^{11})x_t,$$

$$y_{2t} = -(1 - B + B^2 - B^3 + B^4 - B^5 + B^6 - B^7 + B^8 - B^9 + B^{10} - B^{11})x_t,$$

$$y_{3t} = -(B - B^3 + B^5 - B^7 + B^9 - B^{11})x_t,$$

$$y_{4t} = -(1 - B^2 + B^4 - B^6 + B^8 - B^{10})x_t,$$

$$y_{5t} = -\frac{1}{2}(1 + B - 2B^2 + B^3 + B^4 - 2B^5 + B^6 + B^7 - 2B^8 + B^9 + B^{10} - 2B^{11})x_t,$$

$$y_{6t} = \frac{\sqrt{3}}{2}(1 - B + B^3 - B^4 + B^6 - B^7 + B^9 - B^{10})x_t,$$

$$y_{7t} = \frac{1}{2}(1 - B - 2B^2 - B^3 + B^4 + 2B^5 + B^6 - B^7 - 2B^8 - B^9 + B^{10} + 2B^{11})x_t,$$

$$y_{8t} = \frac{\sqrt{3}}{2}(1 + B - B^3 - B^4 + B^6 + B^7 - B^9 - B^{10})x_t,$$

$$y_{9t} = -\frac{1}{2}(\sqrt{3} - B + B^3 - \sqrt{3}B^4 + 2B^5 - \sqrt{3}B^6 + B^7 - B^9 + \sqrt{3}B^{10} - 2B^{11})x_t,$$

$$y_{10t} = \frac{1}{2}(1 - \sqrt{3}B + 2B^2 - \sqrt{3}B^3 + B^4 - B^6 + \sqrt{3}B^7 - 2B^8 + \sqrt{3}B^9 - B^{10})x_t,$$

$$y_{11t} = \frac{1}{2}(\sqrt{3} + B - B^3 - \sqrt{3}B^4 - 2B^5 - \sqrt{3}B^6 - B^7 + B^9 + \sqrt{3}B^{10} + 2B^{11})x_t,$$

$$y_{12t} = \frac{1}{2}(1 + \sqrt{3}B + 2B^2 + \sqrt{3}B^3 + B^4 - B^6 - \sqrt{3}B^7 - 2B^8 - \sqrt{3}B^9 - B^{10})x_t,$$

$$z_t^{12} = (1 - B^{12})x_t.$$

Regressions similar to (3.1) through (3.3), can then be defined as:

$$z_t^{12} = \sum_{i=1}^{12} \pi_{it}\, y_{it-1} + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^{12} + \mu_t \qquad \text{(A.21)}$$

$$z_t^{12} = \sum_{i=1}^{12} \pi_{it}\, y_{it-1} + \mu_t + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^{12} + \mu_t \qquad \text{(A.22)}$$

$$z_t^{12} = \sum_{i=1}^{12} \pi_{it}\, y_{it-1} + \mu_t + \beta_t(n - N/2) + \sum_{j=1}^{p-3} \theta_{jt}\, z_{t-j}^{12} + \mu_t \qquad \text{(A.23)}$$

The hypotheses of interest, test statistics and distributions drawn from these regressions appear in Table 3.1. The hypotheses $H_0^A(12)$ and $H_0^S(12)$ are analogous to the quarterly $H_0^A(4)$ and $H_0^S(4)$ appearing in the main body of the text.

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