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# Efficient Income Redistribution in a Growing Economy

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# Efficient Income Redistribution in a Growing Economy*

*Gerhard Sorger*†

## Résumé / Abstract

On analyse un modèle néoclassique où la population se compose de deux classes (les capitalistes et les ouvriers). Les capitalistes consomment une portion de leur revenu et éprgnent le reste. Les ouvriers consomment tout leur revenu et n'épargnent rien. Le gouvernement peut redistribuer le revenu entre les deux classes au moyen des taxes et des transferts forfaitaires. D'abord on formule un résultat de Kaitala et Pohjola (1990) qui caractérise l'ensemble des solutions optimales de premier rang. Ensuite on démontre que dans le jeu différentiel entre la classe capitaliste et le gouvernement, il existe un équilibre Markov-parfait de Nash qui coïncide avec une solution optimale de premier rang. Cet équilibre implique des transferts à long terme du revenu de la classe ouvrière à la classe capitaliste.

*We consider a neoclassical growth model in which the society consists of two classes (capitalists and workers). The capitalists consume part of their income and save the rest, whereas the workers are assumed to consume their entire income immediately without saving anything. The governement can redistribute income between the two classes by lump-sum transfers and taxation. We first state the result due to Kaital and Pohjola (1990), which characterizes the set of all first best solutions. We then show that the differential game between the capitalistic class and the government has a Markov-perfect Nash equilibrium which coincides with one of the first best solutions. This equilibrium features a long-run transfer of all wage income from the workers to the capitalists.*

**Mots Clés :** Redistribution du revenu, modèle néoclassique de la croissance, jeu différentiel, équilibre Markov-parfait de Nash

**Keywords :** Income redistribution, Neoclassical growth model, Differential game, Markov-perfect Nash equilibrium

JEL : H23, H24

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## 1 Introduction

The study of growth in an economy in which different classes of the society determine their saving rates in different ways goes at least back to Pasinetti (1962). In the simplest setting there are 2 classes which are conveniently referred to as "the capitalists" and "the workers," respectively. Hamada (1967) assumes that the average saving rate of the capitalists is higher than that of the workers (both are assumed to be constant) and shows that, along a balanced optimal growth path, the workers cannot be made better off by any (positive or negative) income transfer between the two classes. He also proves that in the case where the economy starts off the balanced path and workers are impatient, an eventual income transfer from the capitalists to the workers is optimal from the point of view of the working class. The latter result is derived under the assumption that workers do not save at all and that the saving rate of the capitalists is a fixed positive constant. Kaitala and Pohjola (1990) relax the assumption of a fixed saving rate by considering the case where the capitalists make their saving decision in an optimal way. This leads to a differential game model in which player 1 is the class of capitalists and player 2 is the government. The capitalists control the overall saving rate of the economy and the government determines the income redistribution between the two classes. Kaitala and Pohjola (1990) prove that in the case where the players can only use stationary Markovian strategies the equilibrium saving rate is equal to 0 and the capital/labor ratio converges to 0. Needless to say that this equilibrium has very poor welfare properties. Therefore, Kaitala and Pohjola (1990) allow for non-Markovian (trigger) strategies and demonstrate that under this assumption there exist efficient Nash equilibria, in which the long-run capital/labor ratio is determined by the modified golden-rule. The application of trigger strategies implies that the players condition their decisions at any time $t$ on the entire history of the game up to time $t$.

A crucial assumption in the paper by Kaitala and Pohjola (1990) is that the "workers run the government" (p. 425) so that the government's objective is to maximize the present value of consumption by a representative worker. In the present note we modify the model by assuming that the government maximizes the present value of aggregate consumption (by both workers and capitalists). We show that in this case a first best solution is also a Markov-perfect Nash equilibrium of the differential game. In contrast to the situation considered by Kaitala and Pohjola (1990), an efficient outcome of the game exists even if the players use stationary Markovian strategies. These strategies are much simpler than trigger strategies because they depend only on the current state of the system (the capital labor/ratio) and not on the entire history of the game.

We present the model and the results in the Section 2 and conclude the paper with some final remarks in Section 3.

## 2 Model formulation and results

Assume that there are $N^c$ capitalists and $N^w$ workers and that the population grows at a constant rate $\gamma > 0$, i. e., $\dot{N}^c/N^c = \dot{N}^w/N^w = \gamma$.[1] By choosing appropriate units of measurement we may assume that $N^c(0) + N^w(0) = 1$. The (constant) percentage of capitalists in the overall population is denoted by $\lambda = N^c/(N^c + N^w)$. Each worker is endowed with 1 unit of labor per unit of time which he provides inelastically to the production sector. The total labor supply is therefore $N^w$ units per time. Workers do not save and, thus, do not hold any capital. The aggregate capital stock of the economy will be denoted by $K$ and each capitalist holds $K/N^c$ units of it. Aggregate output is given by $F(K, N^w)$ where $F$ is a neoclassical production function with constant returns to scale. We define $f(k) = F(k, 1)$, where $k = K/N^w$ denotes the capital/labor ratio, and assume that $f$ is continuous on $[0, \infty)$, and twice continuously differentiable, increasing, and strictly concave on $(0, \infty)$. Moreover, $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$. Perfect competition implies that the interest rate and the wage rate are given by $f'(k)$ and $f(k) - kf'(k)$, respectively.

A single worker's income is given by $f(k) - kf'(k) + x$, where $x$ denotes the transfer payment (if it is positive) or lump-sum tax (if it is negative) per worker. Since workers do not save, their income equals their consumption. As in Hamada (1967) and Kaitala and Pohjola (1990) we postulate linear utility functions for all agents so that $f(k) - kf'(k) + x$ can also be interpreted as the instantaneous utility derived by a single worker at any given point in time. The present value of utility derived by all members of the working class is therefore given by

$$J^w = (1 - \lambda) \int_0^\infty e^{-rt}[f(k) - kf'(k) + x]\, dt$$

where $r$ denotes the difference between the workers' time preference rate and the population growth rate $\gamma$. It is assumed that $r > 0$.

[1] For simplicity of exposition time arguments are omitted whenever possible. For example, $N^c$ is shorthand of $N^c(t)$.

Denoting the saving rate of the capitalists by $s$ and the lump-sum transfer (or tax) per capitalist by $y$ it follows that the consumption (instantaneous utility) of a single capitalist is given by $(1-s)[(K/N^c)f'(k)+y]$. We assume that the government's budget balances at each point in time which means that $N^w x + N^c y = 0$ or, equivalently, $y = -(N^w/N^c)x$. Therefore, the instantaneous utility of a single capitalist can be written as $[(1-\lambda)/\lambda](1-s)[kf'(k)-x]$. If we assume (as in Kaitala and Pohjola, 1990) that the capitalists have the same time preference rate as the workers, then it follows that the present value of utility derived by all capitalists is given by

$$J^c = (1-\lambda)\int_0^\infty e^{-rt}(1-s)[kf'(k)-x]\,dt.$$

Aggregate investment is equal to aggregate saving so that $\dot{K} = s[Kf'(k) - N^w x]$. Because of the constant population growth this implies that

$$\dot{k} = s[kf'(k)-x] - \gamma k\,, \qquad k(0) = k_0. \tag{1}$$

The capitalists can choose the saving rate within its feasible limits

$$0 \leq s \leq 1 \tag{2}$$

and the government can choose taxes and transfer payments $x$ such that

$$kf'(k) - f(k) \leq x \leq kf'(k). \tag{3}$$

Constraint (3) is necessary for the feasibility of the tax/transfer policy because it ensures that both capitalists and workers have nonnegative consumption at all points in time.

So far, our model is identical to the one studied by Kaitala and Pohjola (1990). We now depart from their setting by assuming that the government's objective is to maximize the present value of aggregate consumption (utility) given by $J^g = J^w + J^c$. Note that $J^w$ and $J^c$ are the utility derived by the working class and the capitalistic class, respectively, so that adding $J^w$ and $J^c$ gives aggregate utility. Using the definition of $J^w$ and $J^c$ it follows that

$$J^g = (1-\lambda)\int_0^\infty e^{-rt}\{f(k) - s[kf'(k)-x]\}\,dt.$$

Before we discuss Markov-perfect Nash equilibria for the differential game between the capitalists and the government we state a result which characterizes the first best solution of the government's optimization problem. By this we mean an optimal solution to the control problem with objective functional $J^g$, controls $x$ and $s$, and constraints (1) - (3). We denote by $\bar{k}$ the golden-rule capital/labor ratio, which is the unique number $\bar{k} > 0$ satisfying the equation $f'(\bar{k}) = r + \gamma$.

**Theorem 1** *(i) If the government can control both the transfer rate $x$ and the saving rate $s$ then the objective functional $J^g$ is maximized subject to (1) - (3) if and only if*

$$s[kf'(k) - x] = \begin{cases} f(k) & \text{if } k < \bar{k}, \\ \gamma \bar{k} & \text{if } k = \bar{k}, \\ 0 & \text{if } k > \bar{k}. \end{cases} \tag{4}$$

*If (4) holds at each point in time then, after some finite time (depending on the initial capital stock), the capital/labor ratio will be constant and equal to $\bar{k}$.*
*(ii) Condition (4) can be satisfied, for example, by choosing the stationary Markovian strategies $s = \sigma(k)$ and $x = \xi(k)$ where*

$$\sigma(k) = \begin{cases} 1 & \text{if } k < \bar{k}, \\ \gamma\bar{k}/f(\bar{k}) & \text{if } k \geq \bar{k}, \end{cases} \quad \text{and} \quad \xi(k) = \begin{cases} kf'(k) - f(k) & \text{if } k \leq \bar{k}, \\ kf'(k) & \text{if } k > \bar{k}. \end{cases}$$

Proof. The first assertion is proved in Kaitala and Pohjola (1990, p. 426) by the same method that will be used in the proof of Theorem 2 below. The second assertion follows immediately by substituting $s = \sigma(k)$ and $x = \xi(k)$ into (4). □

Note that the strategy pair $(\sigma(\cdot), \xi(\cdot))$ is not the only one that satisfies (4) because $s$ and $x$ are uniquely defined by (2) - (4) only for $k \in [0, \bar{k})$. In particular, (4) allows positive, negative, or zero transfers between the two classes if the capital/labor ratio is equal to its long-run steady state $\bar{k}$. The strategy pair $(\sigma(\cdot), \xi(\cdot))$ defined in the theorem entails a complete income transfer from the workers to the capitalists in the long-run and is therefore, from the workers' point of view, a very bad choice among the first best solutions.

Kaitala and Pohjola (1990) consider the differential game in which the government maximizes $J^w$ and the capitalists maximize $J^c$. Assuming that the players use stationary Markovian strategies, they prove that in equilibrium the capitalists do not save ($s = 0$) and the government transfers all income from the capitalists to the workers ($x = kf'(k)$). It can be shown that this outcome is the same as if each player would actually minimize his opponent's objective functional instead of maximizing his own. Thus, the lack of coordination in the model used by Kaitala and Pohjola (1990) leads to the worst outcome for both players. In order to generate a more reasonable prediction from their model, Kaitala and Pohjola (1990) allow for trigger strategies which are informationally more demanding than stationary Markovian strategies because the computation of the controls at time $t$ involves the entire history of the game up to time $t$.[2] They show that, if the players use these more complicated strategies, first best solutions of the model can be supported as Nash equilibria. In the rest of this section we shall show that in our modification of the game, first best solutions can be supported as Nash equilibria even if one restricts the players to use stationary Markovian strategies.

Before we state the main result it is worth mentioning that the stationary Markovian strategies $s = 0$ and $x = kf'(k)$, which constitute the unique Markov-perfect Nash equilibrium in the model of Kaitala and Pohjola (1990), do also form an equilibrium in the present model in which the government is assumed to maximize $J^g$ instead of $J^w$. This follows from the following observations:
(i) if the capitalists choose the saving rate identically equal to 0, the government's control $x$ does not have any influence on the government's optimization problem anymore,
(ii) if the government chooses $x = kf'(k)$, the capitalists' problem becomes independent of their control variable $s$.
Thus, in this equilibrium, both players are indifferent between all possible strategies and might as well choose the equilibrium strategies. However, contrary to the case considered by Kaitala and Pohjola (1990), this trivial equilibrium is not the only Markov-perfect Nash equilibrium. As a matter of fact, the following theorem shows that the first best solution $(\sigma(\cdot), \xi(\cdot))$ defined in Theorem 1 qualifies as a Markov-perfect Nash equilibrium of the differential game in which the government maximizes $J^g$. It follows that history-dependent trigger strategies are not necessary for obtaining a first best solution if one assumes that the government properly takes into account the utility of all agents.

**Theorem 2** *Consider the differential game in which the government chooses the transfers rate $x$ in order to maximize $J^g$ and the capitalists choose the saving rate $s$ so as to maximize $J^c$. The pair $(\sigma(\cdot), \xi(\cdot))$ defined in Theorem 1 is a Markov-perfect Nash equilibrium.*

PROOF. We have to show that $\xi(\cdot)$ is a best response to $\sigma(\cdot)$ and vice

[2] In the case of a stationary Markovian strategy like $s = \sigma(k)$ the control value depends only on the current state of the system.

versa. Assume that the capitalists use the strategy $s = \sigma(k)$. Using Equation (1) and partial integration we see that

$$J^g = (1-\lambda)\int_0^\infty e^{-rt}[f(k) - \dot{k} - \gamma k]\,dt$$
$$= (1-\lambda)\left\{\int_0^\infty e^{-rt}[f(k) - (r+\gamma)k]\,dt + k_0\right\}.$$

The integrand on the right hand side of this equation is a unimodal function of $k$ which attains its unique maximum at $k = \bar{k}$ (see Figure 1a). Therefore, the government will try to steer the capital/labor ratio as close to the value $\bar{k}$ as possible. This means that it will try to satisfy (4) with $s$ replaced by $\sigma(k)$. Because the feedback law $x = \xi(k)$ achieves this goal it is a best response to $\sigma(\cdot)$.

Now assume that the government plays $x = \xi(k)$. We obtain

$$J^c = (1-\lambda)\int_0^\infty e^{-rt}[kf'(k) - \xi(k) - \dot{k} - \gamma k]\,dt$$
$$= (1-\lambda)\left\{\int_0^\infty e^{-rt}[kf'(k) - \xi(k) - (r+\gamma)k]\,dt + k_0\right\}.$$

The integrand of this objective functional can be written as

$$kf'(k) - \xi(k) - (r+\gamma)k = \begin{cases} f(k) - (r+\gamma)k & \text{if } k \leq \bar{k}, \\ -(r+\gamma)k & \text{if } k > \bar{k}. \end{cases}$$

and is depicted in Figure 1b. It is also a unimodal function of $k$ which attains its unique maximum at the value $k = \bar{k}$. Therefore, the capitalists, too, will try to steer the capital/labor ratio as fast as possible to $\bar{k}$, that is, they will try to satisfy (4) with $x$ replaced by $\xi(k)$. Because the feedback law $s = \sigma(k)$ achieves this goal it qualifies as a best response to $\xi(\cdot)$. This completes the proof of the theorem. □

A similar remark as the one made after Theorem 1 applies also to Theorem 2. There are other pairs of stationary Markovian strategies $(\tilde{\sigma}(\cdot), \tilde{\xi}(\cdot))$ which constitute both a first best solution and a Markov-perfect Nash equilibrium. For the proof of Theorem 2 to remain vaild, however, it is necessary that they coincide with $(\sigma(\cdot), \xi(\cdot))$ on the interval $[0, \bar{k}]$. In particular, the specifications $\tilde{\sigma}(\bar{k}) = \gamma\bar{k}/f(\bar{k})$ and $\tilde{\xi}(\bar{k}) = \bar{k}f'(\bar{k}) - f(\bar{k})$ are crucial since without them the integrand of $J^c$ would not attain its maximum at $k = \bar{k}$. This implies that the complete transfer of income from the workers to the capitalists is an essential feature of the Markov-perfect Nash equilibria derived in that way. This conclusion, which is opposite to the one derived by Hamada (1967), should not come as a surprise because we consider a game in which the workers are not even among the players whereas Hamada (1967) studies what is optimal for the workers.

## 3 Concluding remarks

We have reconsidered a model by Kaitala and Pohjola (1990) which describes the strategic interaction between the government and the capitalistic class in a neoclassical growth framework. We have modified that model by assuming that the government maximizes the aggregate utility of all agents (workers and capitalists). Whereas the only Markov-perfect Nash equilibrium of the original model (in which the government is assumed to maximize only the utility of the workers) has extremely poor welfare properties, our model admits Markov-perfect Nash equilibria which coincide with first best solutions. These equilibria result in a complete income transfer from the working class to the capitalists once the efficient golden-rule capital/labor ratio is reached. This feature is probably a consequence of the assumption that the workers in this model do not have any strategic power. A worthwhile project for future research would therefore be to extend the model by introducing a control variable for the working class and deriving equilibria in this 3-player setting. For example, one might think of a model in which the workers (i. e., the union) decides on the labor supply or a model in which each class can force the government out of power if its after tax income is too low as compared to the income of the other class.

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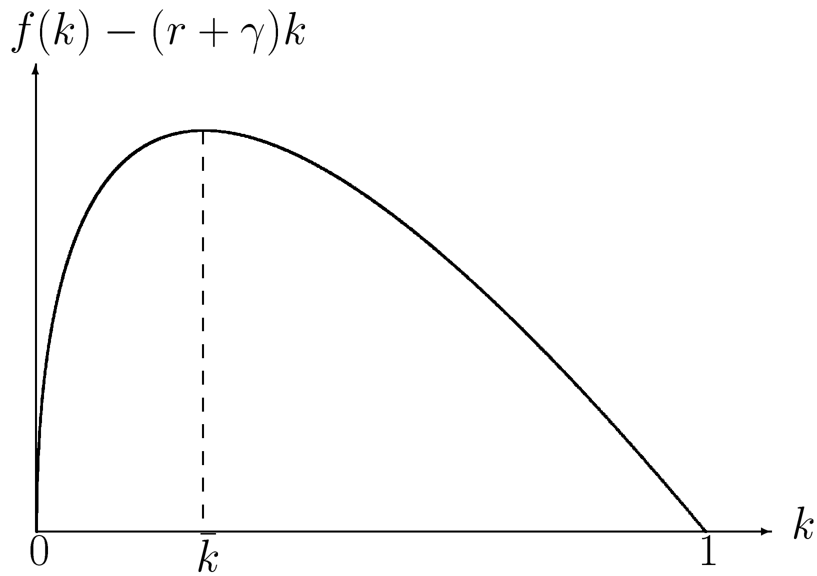


Figure 1a

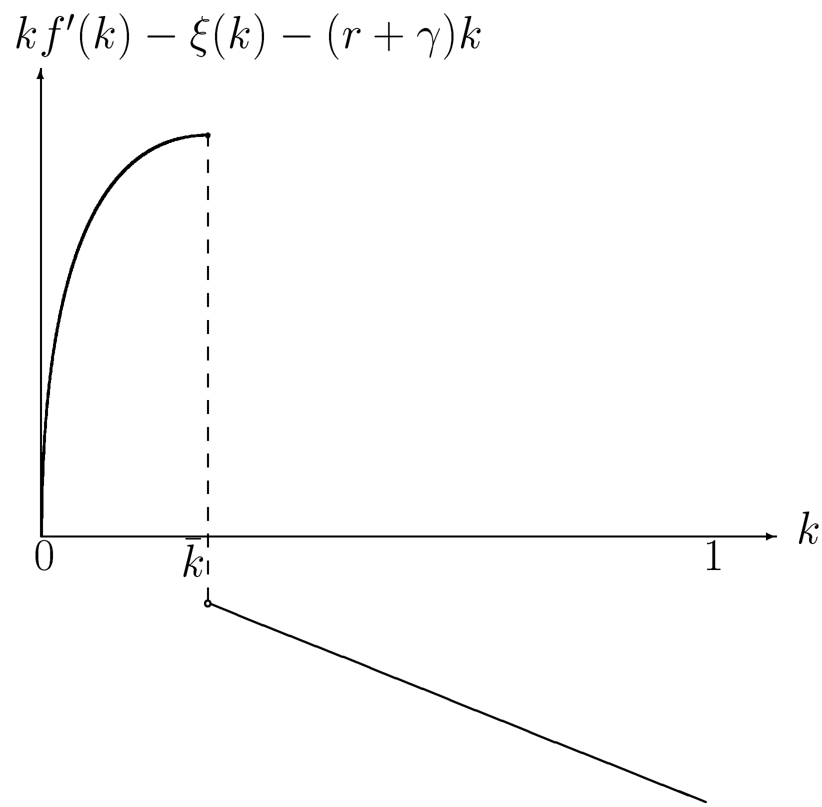


Figure 1b

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