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Résumé / Abstract

Dans cet article, nous étudions les propriétés des processsus avec racines unitaires saisonnières et avec racines quasi-unitaires. Nous traitons le cas des marchés aléatoires ainsi que les processus plus généraux et analysons les distributions des estimateurs et les fonctions de puissances de plusieurs tests.

This paper presents a detailed discussion of the characteristics of seasonal integrated and near integrated processes, as well as the asymptotic properties of seasonal unit root tests. More specifically, the characteristics of a seasonal random walk and a more general seasonal integrated ARMA process are analysed. Also the implications of modelling nonstationary stochastic season-ality as deterministic are highlighted. A further observation made includes the asymptotic distributions and power functions of several seasonal unit root tests.

Mots Clés : Saisonnalité déterministique et stochastique, racines unitaires saisonnières

Keywords : Deterministic/stochastic seasonality, seasonal unit roots

JEL : C13, C22

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1 Introduction

Over the last three decades there has been an increasing interest in modelling seasonality. Progressing from the traditional view that the seasonal pattern is a nuisance which needed to be removed, it is now considered to be an informative feature of economic time series which should be modelled explicitly (see for instance Ghysels (1994) for a review).

Since the seminal work by Box and Jenkins (1970), the stochastic properties of seasonality have been a major focus of research. In particular, the recognition that the seasonal behaviour of economic time series may be varying and changing over time due to the presence of seasonal unit roots (see for example Hylleberg (1994), Hylleberg, Jørgensen and Sørensen (1993) and Osborn (1990)), has led to the development of a considerable number of testing procedures (inter alia, Canova and Hansen (1995), Dickey, Hasza and Fuller (1984), Franses (1994), Hylleberg, Engle, Granger and Yoo (1990) and Osborn, Chui, Smith and Birchenhall (1988)).

In this paper, we review the properties of stochastic seasonal nonstationary processes, as well as the properties of several seasonal unit root tests. More specifically, in Section 2 we analyze the characteristics of the seasonal random walk and generalize our discussion for seasonally integrated ARMA processes. Furthermore, we also illustrate the implications that can emerge when nonstationary stochastic seasonality is posited as deterministic. In Section 3 we consider the asymptotic properties of the seasonal unit root test procedures proposed by Dickey, Hasza and Fuller (1984) and Hylleberg, Engle, Granger and Yoo (1990). Section 4 generalizes most of the results of Section 3 by considering the behavior of the test procedures in a near seasonally integrated framework. Finally, Section 5 concludes the paper.

2 Properties of Seasonal Unit Root Processes

The case of primary interest in the context of seasonal unit roots occurs when the process y_t is nonstationary and annual differencing is required to induce stationarity. This is often referred to as *seasonal integration*. More formally:

> *Definition 2.1: The nonstationary stochastic process y_t, observed at $\mathbb{S}$ equally spaced time intervals per year, is said*

to be seasonally integrated of order d, denoted $y_t \sim SI(d)$, if $\Delta_{\mathbb{S}}^d y_t = (1-L^{\mathbb{S}})^d y_t$ is a stationary, invertible ARMA process.

Therefore, if first order annual differencing renders y_t a stationary and invertible process, then $y_t \sim SI(1)$. The simplest case of such a process is the seasonal random walk, which will be the focus of analysis throughout most of this paper. We refer to $\mathbb{S}$ as the number of seasons per year for y_t.

2.1 The Seasonal Random Walk

The seasonal random walk is a seasonal autoregressive process of order 1, or SAR(1), such that

$$y_t = y_{t-\mathbb{S}} + \varepsilon_t, \qquad t = 1, 2, ..., T \tag{1}$$

with $\varepsilon_t \sim iid(0, \sigma^2)$. Denoting the season in which observation t falls as s_t, with $s_t = 1 + (t-1) \bmod \mathbb{S}$, backward substitution for lagged y_t in this process implies that

$$y_t = y_{s_t - \mathbb{S}} + \sum_{j=0}^{n_t - 1} \varepsilon_{t-\mathbb{S}j} \tag{2}$$

where $n_t = 1 + [(t-1)/\mathbb{S}]$ and $[.]$ represents the greatest integer less or equal to $(t-1)/\mathbb{S}$. As noted by Dickey, Hasza and Fuller (1984) and emphasized by Osborn (1993), the random walk in this case is defined in terms of the disturbances for the specific season s_t only, with the summation over the current disturbance ε_t and the disturbance for this season in the $n_t - 1$ previous years of the observation period. The term $y_{s_t-\mathbb{S}} = y_{t-n_t\mathbb{S}}$, refers to the appropriate starting value for the process. Equation (1) is, of course, a generalization of the conventional nonseasonal random walk.

Note that the unconditional mean of y_t from (2) is

$$E(y_t) = E(y_{s_t-\mathbb{S}}). \tag{3}$$

Thus, although the process (1) does not explicitly contain deterministic seasonal effects, these are implicitly included when $E(y_{s_t-\mathbb{S}})$ is nonzero and varies over $s_t = 1, ..., \mathbb{S}$.

In their analysis of seasonal unit roots, Dickey *et al.* (1984) separate the y_t corresponding to each of the $\mathbb{S}$ seasons into distinct series. Notationally, this is conveniently achieved using two subscripts, the first referring to the season and the second to the year. Then

$$y_t = y_{s+\mathbb{S}(n-1)} = y_{sn} \tag{4}$$

where s_t and n_t are here written as s and n for simplicity of notation. Correspondingly $\mathbb{S}$ disturbance series can be defined as

$$\varepsilon_t = \varepsilon_{s_t+\mathbb{S}(n_t-1)} = \varepsilon_{sn}. \tag{5}$$

Using these definitions, and assuming that observations are available for precisely N $(N = T/\mathbb{S})$ complete years, then (1) can be written as

$$y_{sn} = y_{s,0} + \sum_{j=1}^{n} \varepsilon_{sj} \qquad s = 1, ..., \mathbb{S} \text{ and } n = 1, ..., N \tag{6}$$

which simply defines a random walk for each season $s = 1, ..., \mathbb{S}$.

Because the disturbances ε_t of (1) are uncorrelated, the random walks defined by (6) for the $\mathbb{S}$ seasons of the year are also uncorrelated. Thus, any linear combination of these processes can itself be represented as a random walk. The accumulation of disturbances allows the differences to wander far from the mean over time, giving rise to the phenomenon that "summer may become winter".

2.2 More General Processes

To generalize the above discussion, weakly stationary autocorrelations can be permitted in the $SI(1)$ process. That is, (1) can be generalized to the seasonally integrated ARMA process:

$$\phi(L)\Delta_{\mathbb{S}} y_t = \theta(L)\varepsilon_t, \qquad t = 1, 2, ..., T \tag{7}$$

where, as before, $\varepsilon_t \sim iid(0, \sigma^2)$, while the polynomials $\phi(L)$ and $\theta(L)$ in the lag operator L have all roots outside the unit circle. It is, of course, permissible that these polynomials take the multiplicative form of the seasonal ARMA model of Box and Jenkins (1970).

Inverting the stationary autoregressive polynomial and defining $z_t = \phi(L)^{-1}\theta(L)\ \varepsilon_t$, we can write (7) as:

$$\Delta_{\mathbb{S}} y_t = z_t, \qquad t = 1, ..., T. \tag{8}$$

The process superficially looks like the seasonal random walk, namely (1). There is, however, a crucial difference in that z_t here is a stationary, invertible ARMA process. Nevertheless, performing the same substitution for lagged y_t as above leads to the corresponding result, which can be written as

$$y_{sn} = y_{s,0} + \sum_{j=1}^{n} z_{sj} \qquad s = 1, ..., \mathbb{S} \text{ and } n = 1, ..., N \tag{9}$$

As in (6), (9) implies that there are $\mathbb{S}$ distinct unit root processes, one corresponding to each of the seasons. The important distinction is that these processes in (9) may be autocorrelated and cross-correlated. Nevertheless, it is only the stationary components which are correlated.

Defining the observation and (weakly stationary) disturbance vectors for year n as $Y_n = (y_{1n}, ..., y_{\mathbb{S}n})'$ and $Z_n = (z_{1n}, ..., z_{\mathbb{S}n})'$ respectively, the vector representation of (9) is:

$$\Delta Y_n = Z_n, \qquad n = 1, ..., N. \tag{10}$$

The disturbances here follow a stationary vector ARMA process

$$\Phi(L) Z_n = \Theta(L) E_n. \tag{11}$$

It is sufficient to note that $\Phi(L)$ and $\Theta(L)$ are appropriately defined $\mathbb{S} \times \mathbb{S}$ polynomial matrices in L with all roots outside the unit circle and $E_n = (\varepsilon_{1n}, ..., \varepsilon_{\mathbb{S}n})'$. The seasonal difference of (7) is converted to a first difference in (10) because $\Delta Y_n = Y_n - Y_{n-1}$ defines an annual (that is, seasonal) difference of the vector Y_t.

Now, in (10) we have a vector ARMA process in ΔY_n, which is a vector ARIMA process in Y_n. In the terminology of Engle and Granger (1987), the $\mathbb{S}$ processes in the vector Y_t cannot be cointegrated if this is the data generating process (DGP). Expressed in a slightly different way, if the process is written in terms of the level Y_n, the vector process will contain $\mathbb{S}$ unit roots due to the presence of the factor $\Delta = 1 - L$ in each of the equations. Therefore, the implication drawn from the seasonal random walk of (1) that any linear combination of the separate seasonal series is itself an $I(1)$ process carries over to this case too.

For the purpose of this paper, only the simple seasonal random walk case will be considered in the subsequent analysis. It should, however, be recognized that the key results extend to more general seasonally integrated processes.

2.3 Asymptotic Properties

Consider the DGP of the seasonal random walk with initial values $y_{-\mathbb{S}+s} = ... = y_0 = 0$. Using the notation of (6), the following $\mathbb{S}$ independent partial sum processes (PSPs) can be obtained:

$$S_{sn} = \sum_{j=1}^{n} \varepsilon_{sj} \qquad s = 1, ..., \mathbb{S}, n = 1, ..., N \tag{12}$$

where n represents the number of years of observations to time t. From the Functional Central Limit Theorem (FCLT) and the Continuous Mapping Theorem (CMT) the appropriately scaled PSPs in (12) converge as $N \to \infty$ to

$$\frac{1}{\sqrt{N}} S_{sn} \Rightarrow \sigma W_s(r) \tag{13}$$

where $\Rightarrow$ indicates convergence in distribution, while $W_s(r)$, $s = 1, ..., \mathbb{S}$ are independent standard Brownian motions.

Furthermore, the following Lemma collecting the relevant convergence results for seasonal unit root processes of periodicity $\mathbb{S}$ can be stated:

Lemma 1 *Assuming that the DGP is the seasonal random walk in (1) with initial values equal to zero, $\varepsilon_t \sim iid(0, \sigma^2)$ and $T = \mathbb{S}N$, then from the CMT, as $T \to \infty$,*

a) $T^{-1/2} y_{t-k} \Rightarrow \mathbb{S}^{-1/2} \sigma L^k W_s$

b) $T^{-3/2} \sum_{t=1}^{T} y_{t-k} \Rightarrow \mathbb{S}^{-3/2} \sigma \sum_{s=1}^{\mathbb{S}} \int_0^1 W_s dr$

c) $T^{-2} \sum_{t=1}^{T} y_{t-i} y_{t-k} \Rightarrow \mathbb{S}^{-2} \sigma^2 \sum_{s=1}^{\mathbb{S}} \int_0^1 W_s (L^{k-i} W_s) dr \qquad k \geq i$

d) $T^{-1} \sum_{t=1}^{T} y_{t-k} \varepsilon_t \Rightarrow \mathbb{S}^{-1} \sigma^2 \sum_{s=1}^{\mathbb{S}} \int_0^1 (L^k W_s) dW_s$

where $k = 1, ..., \mathbb{S}$, $W_s(r)$ ($s = t \bmod \mathbb{S}$ and $s = \mathbb{S}$ when $t \bmod \mathbb{S} = 0$) are independent standard Brownian motions, L is the lag operator which shifts the Brownian motions between seasons ($L^k W_s = W_{s-k}$ with $W_{s-k} = W_{\mathbb{S}+s-k}$ for $s - k \leq 0$) and $W_s = W_s(r)$ for simplicity of notation.

It is important to note the circular property regarding the rotation of the W_k, so that after $\mathbb{S}$ lags of y_t the same sum of $\mathbb{S}$ integrals emerges. The Lemma is established in Osborn and Rodrigues (1998).

2.4 Deterministic Seasonality

A common practice is to attempt the removal of seasonal patterns via seasonal dummy variables (see, for example, Barsky and Miron (1989), Beaulieu and Miron (1991), Osborn (1990)). The interpretation of the seasonal dummy approach is that seasonality is essentially deterministic so that the series is stationary around seasonally varying means. The simplest deterministic seasonal model is

$$y_t = \sum_{s=1}^{\mathbb{S}} \delta_{st} m_s + \varepsilon_t \tag{14}$$

where δ_{st} is the seasonal dummy variable which takes the value 1 when t falls in season s and $\varepsilon_t \sim iid(0, \sigma^2)$. Typically, y_t is a first difference series in order to account for the zero frequency unit root commonly found in economic time series.

When a model like (14) is used, the coefficient of determination (R^2) is often computed as a measure of the strength of the seasonal pattern. However, as Abeysingh (1991, 1994) and Franses, Hylleberg and Lee (1995) indicate, the presence of seasonal unit roots in the DGP will have important consequences for R^2.

To illustrate this issue, take the seasonal random walk of (1) as the DGP and assume that (14) is used to model the seasonal pattern. As is well known, the OLS estimates of m_s, $s = 1, ..., \mathbb{S}$ are simply the mean values of y_t in each season. Thus, using the notation of (4),

$$\widehat{m}_s = \frac{1}{N} \sum_{t=1}^{T} \delta_{st} y_t = \frac{1}{N} \sum_{t=1}^{N} y_{sn} \tag{15}$$

where (as before) T and N are the total number of observations and the total number of complete years of observations available, respectively and it is again assumed for simplicity that $T = \mathbb{S}N$. As noted by Franses *et al.*, the estimated seasonal intercepts diverge under the seasonal random walk DGP. In particular, the appropriately scaled $\widehat{m}_s$ converge to a normal random variable

$$N^{-1/2}\widehat{m}_s = N^{-3/2} \sum_{t=1}^{T} \delta_{st} y_t \Rightarrow \sigma \int_0^1 W_s(r) dr = N\left(0, \sigma^2/3\right), \; s = 1, ..., \mathbb{S}. \tag{16}$$

where the latter follows from Banerjee et al. (1993, pp. 43-45) who show that $\int_0^1 W(r)dr = N(0, 1/3)$.

For this DGP, the R^2 from (14) has a non-degenerate asymptotic distribution. As shown in the Appendix,

$$R^2 = \frac{\sum_{t=1}^{T} \left(\widehat{y}_t - \overline{y}\right)^2}{\sum_{t=1}^{T} \left(y_t - \overline{y}\right)^2} \Rightarrow \frac{\sum_{s=1}^{\mathbb{S}} \left(\int_0^1 W_s(r)dr\right)^2 - \frac{1}{\mathbb{S}} \left[\int_0^1 \left(\sum_{s=1}^{\mathbb{S}} W_s(r)\right) dr\right]^2}{\sum_{s=1}^{\mathbb{S}} \int_0^1 W_s^2(r)dr - \frac{1}{\mathbb{S}} \left[\int_0^1 \left(\sum_{s=1}^{\mathbb{S}} W_s(r)\right) dr\right]^2}. \tag{17}$$

Consequently, high values for this statistic are to be anticipated, as concluded by Franses *et al*. These are spurious in the sense that the DGP contains no deterministic seasonality since $E(y_t) = 0$ when the starting values for (1) are zero. Hence high a value of R^2 when (14) is estimated does not constitute evidence in favour of deterministic seasonality.

3 Testing the Seasonal Unit Root Null Hypothesis

In this section we discuss the test procedures proposed by Dickey, Hasza and Fuller (1984) and Hylleberg, Engle, Granger and Yoo [HEGY] (1990) to test the null hypothesis of seasonal integration. It should be noted that while there are a large number of seasonal unit root tests available (see, for example, Rodrigues (1998a) for an extensive survey), casual observation of the literature shows that the HEGY test is the most frequently used procedure in empirical work.

For simplicity of presentation, throughout this section we assume that augmentation of the test regression to account for autocorrelation is unnecessary and that pre-sample starting values for the DGP are equal to zero.

3.1 The Dickey-Hasza-Fuller Test

The first test of the null hypothesis $y_t \sim SI(1)$ was proposed by Dickey, Hasza and Fuller [DHF] (1984), as a direct generalization of the test proposed by Dickey and Fuller (1979) for a nonseasonal AR(1) process. Assuming that the process is known to be a $SAR(1)$, then the DHF test can be parameterized as

$$\Delta_{\mathbb{S}} y_t = \alpha_{\mathbb{S}} y_{t-\mathbb{S}} + \varepsilon_t. \tag{18}$$

The null hypothesis of seasonal integration corresponds to $\alpha_{\mathbb{S}} = 0$, while the alternative of a stationary stochastic seasonal process implies $\alpha_{\mathbb{S}} < 0$. The appropriately scaled least squares bias obtained from the estimation of $\alpha_{\mathbb{S}}$ under the null hypothesis is

$$T\widehat{\alpha}_{\mathbb{S}} = \frac{\frac{1}{T}\sum_{t=1}^{T} y_{t-\mathbb{S}}\varepsilon_t}{\frac{1}{T^2}\sum_{t=1}^{T} y_{t-\mathbb{S}}^2} \tag{19}$$

and the associated t-statistic is

$$t_{\widehat{\alpha}_{\mathbb{S}}} = \frac{\frac{1}{T}\sum_{t=1}^{T} y_{t-\mathbb{S}}\varepsilon_t}{\widetilde{\sigma}\left[\frac{1}{T^2}\sum_{t=1}^{T} y_{t-\mathbb{S}}^2\right]^{\frac{1}{2}}} \tag{20}$$

where $\widetilde{\sigma}$ is the usual degrees of freedom corrected estimator of σ. Similarly to the usual Dickey-Fuller approach, the test is typically implemented using (20).

Using the results in c) and d) of Lemma 1, it is straightforward to establish that (19) and (20) converge to

$$\frac{T}{\mathbb{S}}\widehat{\alpha}_{\mathbb{S}} \Rightarrow \frac{\sum_{s=1}^{\mathbb{S}} \int_0^1 W_s(r)dW_s(r)}{\sum_{s=1}^{\mathbb{S}} \int_0^1 W_s^2(r)dr} \tag{21}$$

and

$$t_{\widehat{\alpha}_S} \Rightarrow \frac{\sum_{s=1}^{\mathbb{S}} \int_0^1 W_s(r)dW_s(r)}{\left[\sum_{s=1}^{\mathbb{S}} \int_0^1 W_s^2(r)dr\right]^{\frac{1}{2}}}, \tag{22}$$

respectively. Note that $\widetilde{\sigma}^2 \xrightarrow{p} \sigma^2$.

The asymptotic distribution of the DHF statistic given by (22) is non-standard, but is of similar type to the Dickey-Fuller t-distribution. Indeed, it is precisely the Dickey-Fuller t-distribution in the special case $\mathbb{S} = 1$, when the test regression (18) is the usual Dickey-Fuller test regression for a conventional random walk. It can also be seen from (22) that the distribution for the DHF t-statistic depends on $\mathbb{S}$, that is on the frequency with which observations are made within each year. On the basis of Monte Carlo simulations, DHF tabulated critical values of $\frac{T}{\mathbb{S}}\widehat{\alpha}_{\mathbb{S}}$ and $t_{\widehat{\alpha}_{\mathbb{S}}}$ for various T and $\mathbb{S}$. Note that the limit distributions presented as functions of Brownian motions can also be found in Chan (1989), Boswijk and Franses (1996) and more recently in Osborn and Rodrigues (1998).

To explore the dependence on $\mathbb{S}$ a little further, note first that

$$\int_0^1 W_s(r)dW_s(r) = \frac{1}{2}\left\{[W_s(1)]^2 - 1\right\} \tag{23}$$

where $[W_s(1)]^2$ is $\chi^2(1)$ (see, for example, Banerjee *et al.*, 1993, p.91). The numerator of (22) involves the sum of $\mathbb{S}$ such terms which are mutually independent and hence

$$\begin{aligned} \sum_{s=1}^{\mathbb{S}} \int_0^1 W_s(r)dW_s(r) &= \tfrac{1}{2}\sum_{s=1}^{\mathbb{S}} \left\{[W_s(1)]^2 - 1\right\} \\ &= \tfrac{1}{2}\{\chi^2(\mathbb{S}) - \mathbb{S}\} \end{aligned} \qquad (24)$$

which is half the difference between a $\chi^2(\mathbb{S})$ statistic and its mean of $\mathbb{S}$.

It is well known that the Dickey-Fuller t-statistic is not symmetric about zero. Indeed, Fuller (1976, p.370) comments that asymptotically the probability of (in our notation) $\widehat{\alpha}_1 < 0$ is 0.68 for the nonseasonal random walk because $\Pr[\chi^2(1) < 1] = 0.68$. In terms of (22), the denominator is always positive and hence $\Pr[\chi^2(\mathbb{S}) < \mathbb{S}]$ dictates the probability that $t_{\widehat{\alpha}_{\mathbb{S}}}$ is negative. With a seasonal random walk and quarterly data, $\Pr[\chi^2(4) < 4] = 0.59$, while in the monthly case $\Pr[\chi^2(12) < 12] = 0.55$. Therefore, the preponderance of negative test statistics is expected to decrease as $\mathbb{S}$ increases. As seen from the percentiles tabulated by DHF, the dispersion of $t_{\widehat{\alpha}_{\mathbb{S}}}$ is effectively invariant to $\mathbb{S}$, so that the principal effect of an increasing frequency of observation is a reduction in the asymmetry of this test statistic around zero.

3.2 Testing Complex Unit Roots

Before proceeding to the examination of the procedure proposed by Hylleberg *et al.* (1990) it will be useful to consider some of the issues related to testing complex unit roots, because these are an intrinsic part of any $SI(1)$ process.

The simplest process which contains a pair of complex unit roots is

$$y_t = -y_{t-2} + u_t \qquad (25)$$

with $u_t \sim iid(o, \sigma^2)$. This process has $\mathbb{S} = 2$ and, using the notation identifying the season s and year n, it can be equivalently written as

$$y_{sn} = -y_{s,n-1} + u_{sn} \qquad s = 1, 2 \qquad (26)$$

Notice that the seasonal patterns reverse each year. Due to this alternating pattern, and assuming $y_0 = y_{-1} = 0$, it can be seen that

$$y_t = S^*_{sn} = \sum_{i=0}^{n-1}(-1)^i u_{s,n-i} = -S^*_{s,n-1} + u_{sn} \qquad (27)$$

where, in this case, $n = \left[\frac{t+1}{2}\right]$). Note that S^*_{sn} $(s = 1, 2)$ are independent processes, one corresponding to each of the two seasons of the year. Nevertheless, the nature of the seasonality implied by (25) is not of the conventional type in that S^*_{sj} (for given s) tends to oscillate as j increases. Moreover, it can be observed from (27) that aggregation of the process over full cycles of two years annihilates the nonstationarity as $S^*_{s,n-1} + S^*_{sn} = u_{sn}$. To relate these S^*_{sn} to the $\mathbb{S}$ independent random walks of (6), let $\varepsilon_{sj} = (-1)^j u_{sj}$ which (providing the distribution of u_t is symmetric) has identical properties. Then

$$S^*_{sn} = \begin{array}{ll} \sum_{j=1}^{n} (-1)^{j+1} u_{sj} = -\sum_{j=1}^{n} \varepsilon_{sj} = -S_{jn} & n \text{ odd} \\ \sum_{j=1}^{n} (-1)^{j} u_{sj} = \sum_{j=1}^{n} \varepsilon_{sj} = S_{jn} & n \text{ even} \end{array} \tag{28}$$

where S_{jn} is defined in (12).

Analogously to the DHF test, the unit root process (25) may be tested through the t-ratio for $\widehat{\alpha}^*_2$ in

$$(1 + L^2) y_t = \alpha^*_2 y_{t-2} + u_t. \tag{29}$$

The null hypothesis is $\alpha^*_2 = 0$ with the alternative of stationarity implying $\alpha^*_2 > 0$. Then, assuming $T = 2N$, under the null hypothesis

$$T\widehat{\alpha}^*_2 = \frac{T^{-1} \sum_{t=1}^{T} y_{t-2} u_t}{T^{-2} \sum_{t=1}^{T} y^2_{t-2}} = \frac{(2N)^{-1} \sum_{s=1}^{2} \sum_{j=1}^{N} S^*_{s,j-1}(S^*_{s,j} + S^*_{s,j-1})}{(2N)^{-2} \sum_{s=1}^{2} \sum_{j=1}^{N} (S^*_{s,j-1})^2}. \tag{30}$$

and

$$t(\widehat{\alpha}^*_2) = \frac{\sum_{t=1}^{T} y_{t-2} u_t}{\widetilde{\sigma} \left[\sum_{t=1}^{T} y^2_{t-2}\right]^{\frac{1}{2}}} = \frac{(2N)^{-1} \sum_{s=1}^{2} \sum_{j=1}^{N} S^*_{s,j-1}(S^*_{s,j} + S^*_{s,j-1})}{\widetilde{\sigma} \left[(2N)^{-2} \sum_{s=1}^{2} \sum_{j=1}^{N} (S^*_{s,j-1})^2\right]^{\frac{1}{2}}}. \tag{31}$$

If, for further expositional clarity, we assume that N is even, then using (28), we have

$$
\begin{aligned}
\sum_{j=1}^{N} S^*_{s,j-1}(S^*_{s,j} + S^*_{s,j-1}) &= \sum_{i=1}^{N/2} [S^*_{s,2i-2}(S^*_{s,2i-1} \\
&\quad + S^*_{s,2i-2}) + S^*_{s,2i-1}(S^*_{s,2i} + S^*_{s,2i-1})] \\
&= \sum_{i=1}^{N/2} [S_{s,2i-2}(-S_{s,2i-1} + S_{s,2i-2}) \\
&\quad - S_{s,2i-1}(S_{s,2i} - S_{s,2i-1})] \\
&= -\sum_{j=1}^{N} S_{s,j-1}(S_{s,j} - S_{s,j-1})
\end{aligned}
$$

Thus, there is a "mirror image" relationship between the numerator of (30) and (31) compared with that of (19) and (20) with $\mathbb{S} = 2$. The corresponding denominators are identical as $(S^*_{sj})^2 = S^2_{sj}$. Thus, by applying similar arguments as in the proof of Lemma 1,

$$
\frac{T}{2}\widehat{\alpha}^*_2 \Rightarrow -\frac{\sum_{s=1}^{2} \int_0^1 W_s(r) dW_s(r)}{\sum_{s=1}^{2} \int_0^1 [W_s(r)]^2 dr} \tag{32}
$$

and

$$
t_{\widehat{\alpha}^*_2} \Rightarrow -\frac{\sum_{s=1}^{2} \int_0^1 W_s(r) dW_s(r)}{\left\{\sum_{s=1}^{2} \int_0^1 [W_s(r)]^2 dr\right\}^{\frac{1}{2}}} \tag{33}
$$

which can be compared with (21) and (22) respectively. This mirror image property of these test statistics has also been shown by Fuller (1976, pp.370-372) and Chan and Wei (1988). One important practical consequence of (33) is that with a simple change of sign, the DHF tables with $S = 2$ apply to the case of testing $\alpha^*_2 = 0$ in (29).

Under the assumed DGP (25), we may also consider testing the null hypothesis $\alpha^*_1 = 0$ against the alternative $\alpha^*_1 \neq 0$ in

$$
(1 + L^2)y_t = \alpha^*_1 y_{t-1} + u_t. \tag{34}
$$

The test here is not, strictly speaking, a unit root test, since the unit coefficient on L^2 in (34) implies that the process contains two roots of modulus one, irrespective of the value of α^*_1. Rather, the test of $\alpha^*_1 = 0$ is a test of the null hypothesis that the process contains a half-cycle every $\mathbb{S} = 2$ periods, and hence a full cycle every four periods. The appropriate alternative hypothesis is, therefore, two-sided.

For this test regression,

$$T\widehat{\alpha}_1^* = \frac{T^{-1}\sum\limits_{t=1}^{T} y_{t-1}u_t}{T^{-2}\sum\limits_{t=1}^{T} y_{t-1}^2}.$$

Again referring to (27) and (28), we can see that

$$T\widehat{\alpha}_1^* = \frac{(2N)^{-1}\sum\limits_{j=1}^{N}\left[-S_{2,j-1}(S_{1,j}-S_{1,j-1})+S_{1,j}(S_{2,j}-S_{2,j-1})\right]}{(2N)^{-2}\sum\limits_{j=1}^{N}\left(S_{1,j-1}^2+S_{2,j}^2\right)}. \quad (35)$$

Thus, (35) converges to,

$$\frac{T}{2}\widehat{\alpha}_1^* \Rightarrow \frac{\int_0^1 W_1(r)dW_2(r) - \int_0^1 W_2(r)dW_1(r)}{\sum\limits_{s=1}^{2}\int_0^1 [W_s(r)]^2 dr} \quad (36)$$

and consequently,

$$t_{\widehat{\alpha}_1^*} \Rightarrow \frac{\int_0^1 W_1(r)dW_2(r) - \int_0^1 W_2(r)dW_1(r)}{\left\{\sum\limits_{s=1}^{2}\int_0^1 [W_s(r)]^2 dr\right\}^{\frac{1}{2}}}. \quad (37)$$

Indeed, the results for the distributions associated with the test statistics in (29) and (34) continue to apply for the test regression

$$(1+L^2)y_t = \alpha_1^* y_{t-1} + \alpha_2^* y_{t-2} + \varepsilon_t \quad (38)$$

because the regressors y_{t-1} and y_{t-2} can be shown to be asymptotically orthogonal (see for instance, Ahtola and Tiao (1987) or Chan and Wei (1988) for more details).

3.3 The Hylleberg-Engle-Granger-Yoo Test

It is well known, that the seasonal difference operator $\Delta_{\mathbb{S}} = 1 - L^{\mathbb{S}}$ can always be factorized as

$$1 - L^{\mathbb{S}} = (1-L)(1+L+L^2+...+L^{\mathbb{S}-1}). \quad (39)$$

Hence, (39) indicates that an $SI(1)$ process always contains a conventional unit root and a set of $\mathbb{S}-1$ seasonal unit roots. The approach suggested by Hylleberg, Engle, Granger and Yoo (1990), commonly known as HEGY, examines the validity of $\Delta_{\mathbb{S}}$ through exploiting (39) by testing the unit root of 1 and the $\mathbb{S}-1$ separate nonstationary roots on the unit circle implied by $1+L+...+L^{\mathbb{S}-1}$.

To see the implications of this factorization, consider the case of quarterly data ($\mathbb{S}=4$) where

$$\begin{aligned} 1-L^4 &= (1-L)(1+L+L^2+L^3) \\ &= (1-L)(1+L)(1+L^2). \end{aligned} \tag{40}$$

Thus, $\Delta_4 = 1-L^4$ has four roots on the unit circle[1], namely 1 and -1 which occur at the 0 and π frequencies respectively, and the complex pair $\pm i$ at the frequencies $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Hence, in addition to the conventional unit root, the quarterly case implies three seasonal unit roots, which are -1 and the complex pair $\pm i$. Corresponding to each of the three factors of (40), using a Lagrange approximation, HEGY suggest the following linear transformations:

$$y_{(1),t} = (1+L)(1+L^2)y_t = y_t + y_{t-1} + y_{t-2} + y_{t-3} \tag{41}$$

$$y_{(2),t} = -(1-L)(1+L^2)y_t = -y_t + y_{t-1} - y_{t-2} + y_{t-3} \tag{42}$$

$$y_{(3),t} = -(1-L)(1+L)y_t = -y_t + y_{t-2} \tag{43}$$

By construction, each of the variables in (41) to (43) accepts all the factors of Δ_4 except one. That is, $y_{(1),t}$ assumes the factors $(1+L)$ and $(1+L^2)$, $y_{(2),t}$ assumes $(1-L)$ and $(1+L^2)$, while $y_{(3),t}$ assumes $(1-L)$ and $(1+L)$.

The test regression for quarterly data suggested by HEGY has the form:

$$\Delta_4 y_t = \pi_1 y_{(1),t-1} + \pi_2 y_{(2),t-1} + \pi_3 y_{(3),t-2} + \pi_4 y_{(3),t-1} + \varepsilon_t, \quad t = 1,2,...,T \tag{44}$$

[1] Notice that the unit roots of a monthly seasonal random walk are:

$$1, -1, \pm i, -\tfrac{1}{2}(1 \pm \sqrt{3}i), \tfrac{1}{2}(1 \pm \sqrt{3}i), -\tfrac{1}{2}(\sqrt{3} \pm i), \tfrac{1}{2}(\sqrt{3} \pm i).$$

The first is, once again, the conventional nonseasonal, or zero frequency, unit root. The remaining 11 seasonal unit roots arise from the seasonal summation operator $1+L+L^2+...+L^{11}$ and result in nonstationary cycles with a maximum duration of one year. As can be observed, this monthly case implies five pairs of complex roots on the unit circle.

where $y_{(1),t}, y_{(2),t}$ and $y_{(3),t}$ are defined in (41), (42) and (43), respectively. Note that these regressors are asymptotically orthogonal by construction. The two lags of $y_{(3),t}$ arise because the pair of complex roots $\pm i$ imply two restrictions on a second order polynomial $1 + \phi_1 L + \phi_2 L^2$, namely $\phi_1 = 0$ and $\phi_2 = 1$ (see Section 3.2). The overall null hypothesis $y_t \sim SI(1)$ implies $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ and hence $\Delta_4 y_t = \varepsilon_t$ as for the DHF test.

The HEGY regression (44) and the associated asymptotic distributions can be motivated by considering the three factors of $\Delta_4 = (1-L)(1+L)(1+L^2)$ one by one. Through the variable $y_{(1),t}$, we may consider the DGP

$$y_{(1),t} = y_{(1),t-1} + \varepsilon_t. \tag{45}$$

which is the seasonal random walk of (1) with $\mathbb{S} = 4$ after applying the linear transformation (41). Therefore, when $y_t \sim SI(1)$, $y_{(1),t}$ has the properties of a conventional random walk process and hence, with initial values equal to zero,

$$y_{(1),t} = \sum_{j=0}^{t-1} \varepsilon_{t-j}. \tag{46}$$

Since $\Delta_1 y_{(1),t} = \Delta_4 y_t$, the Dickey-Fuller test regression for the DGP (45) is

$$\Delta_4 y_t = \pi_1 y_{(1),t-1} + \varepsilon_t \tag{47}$$

where we test $\pi_1 = 0$ against $\pi_1 < 0$. Considering

$$T\widehat{\pi}_1 = \frac{T^{-1}\sum_{t=1}^{T} y_{(1),t-1}\varepsilon_t}{T^{-2}\sum_{t=1}^{T} y_{(1),t-1}^2} = \frac{T^{-1}\sum_{t=1}^{T}(y_{t-1}+y_{t-2}+y_{t-3}+y_{t-4})\,\varepsilon_t}{T^{-2}\sum_{t=1}^{T}(y_{t-1}+y_{t-2}+y_{t-3}+y_{t-4})^2} \tag{48}$$

then from Lemma 1 and (13) it can be observed that under the seasonal integration null hypothesis

$$T^{-1}\sum_{t=1}^{T}(y_{t-1}+y_{t-2}+y_{t-3}+y_{t-4})\,\varepsilon_t \Rightarrow \frac{\sigma^2}{4}\left\{\int_0^1 W_{(1)}(r)dW_{(1)}(r)\right\} \tag{49}$$

and

$$T^{-2}\sum_{t=1}^{T}(y_{t-1}+y_{t-2}+y_{t-3}+y_{t-4})^2 \Rightarrow \frac{\sigma^2}{16}\int_0^1 4W_{(1)}^2(r)dr \tag{50}$$

where $W_{(1)}(r) = \sum_{s=1}^{4} W_s(r)$. Substituting (49) and (50) into (48) gives,

$$T\widehat{\pi}_1 \Rightarrow \frac{\int_0^1 W_{(1)}(r)dW_{(1)}(r)}{\int_0^1 \left[W_{(1)}(r)\right]^2 dr} \tag{51}$$

The associated t-statistic, which is commonly used to test for the zero frequency unit root, can be expressed as

$$t_{\widehat{\pi}_1} \Rightarrow \frac{\int_0^1 W_1^*(r)dW_1^*(r)}{\left\{\int_0^1 \left[W_1^*(r)\right]^2 dr\right\}^{\frac{1}{2}}}. \tag{52}$$

where $W_1^*(r) = W_{(1)}(r)/2$. Division by 2 is undertaken here so that $W_1^*(r)$ is standard Brownian motion, whereas $W_{(1)}(r)$ is not. Therefore, (52) is the conventional Dickey-Fuller t-distribution, tabulated by Fuller (1976).

Similarly, based on (42) it can be seen that the seasonal random walk (1) implies

$$-(1+L)y_{(2),t} = \varepsilon_t. \tag{53}$$

Notice the "bounce back" effect in (53) which implies a half cycle for $y_{(2),t}$ every period and hence a full cycle every two periods. Also note that (53) effectively has the same form as (26). Testing the root of -1 implied by (53) leads to a test of $\phi_2 = 1$ against $\phi_2 < 1$ in

$$-(1+\phi_2 L)y_{(2),t} = \varepsilon_t.$$

Equivalently, defining $\pi_2 = \phi_2 - 1$ and again using (42) yields

$$\Delta_4 y_t = \pi_2 y_{(2),t-1} + \varepsilon_t \tag{54}$$

with null and alternative hypotheses $\pi_2 = 0$ and $\pi_2 < 0$, respectively.

Under the null hypothesis, and using analogous reasoning to Section 3.2 combined with Lemma 1, we obtain

$$T\widehat{\pi}_2 \Rightarrow \frac{\int_0^1 W_{(2)}(r)dW_{(2)}(r)}{\int_0^1 \left[W_{(2)}(r)\right]^2 dr} \tag{55}$$

and

$$t_{\widehat{\pi}_2} \Rightarrow \frac{\int_0^1 W_2^*(r)dW_2^*(r)}{\left\{\int_0^1 \left[W_2^*(r)\right]^2 dr\right\}^{\frac{1}{2}}} \tag{56}$$

where the Brownian motion $W_{(2)}(r) = W_1(r) - W_2(r) + W_3(r) - W_4(r)$ is standardized as $W_2^*(r) = W_{(2)}(r)/2$. Like (52), (56) is the conventional Dickey-Fuller distribution tabulated by Fuller (1976). It is important to note that, as indicated by Fuller (1976) and Chan and Wei (1988), the distributions of the least squares bias and corresponding t-statistic when the DGP is an AR(1) with a -1 root are the "mirror image" of those obtained when testing the conventional random walk. However, in (55) and (56), this mirror image is incorporated through the design of the HEGY test regression in that the linear transformation of $y_{(2),t}$ is defined with a minus sign as $-(1-L)(1+L^2)$.

Finally, from (43) it follows that $y_t \sim SI(1)$ implies

$$-(1+L^2)y_{(3),t} = \varepsilon_t. \tag{57}$$

This process implies a "bounce back" after two periods and a full cycle after four. This process has the complex root form identical to (25). Hence, the results presented for that process carry over directly for this case. Noting again that $-(1+L^2)y_{(3),t} = \Delta_4 y_t$, we can test $\phi_3 = 1$ and $\phi_4 = 0$ in

$$-(1+\phi_4 L + \phi_3 L^2)y_{(3),t} = \varepsilon_t$$

through the regression

$$\Delta_4 y_t = \pi_3 y_{(3),t-2} + \pi_4 y_{(3),t-1} + \varepsilon_t \tag{58}$$

with $\pi_3 = \phi_3 - 1$ and $\pi_4 = -\phi_4$. Testing against stationarity implies null and alternative hypotheses of $\pi_3 = 0$ and $\pi_3 < 0$. However, while $\pi_4 = 0$ is also indicated under the null hypothesis, the alternative here is $\pi_4 \neq 0$. The reasoning for this two-sided alternative is precisely that for the test regression (34) and (58) has the same form as (38). Therefore, using similar arguments to Section 3.2, and noting that the "mirror image" property discussed there is incorporated through the minus sign in the definition of $y_{(3),t}$, it can be seen that

$$t_{\widehat{\pi}_3} \Rightarrow \frac{\int_0^1 W_3^*(r)dW_3^*(r) + \int_0^1 W_4^*(r)dW_4^*(r)}{\left\{\int_0^1 [W_3^*(r)]^2\,dr + \int_0^1 [W_4^*(r)]^2\,dr\right\}^{\frac{1}{2}}}. \tag{59}$$

and

$$t_{\widehat{\pi}_4} \Rightarrow \frac{\int_0^1 W_4^*(r)dW_3^*(r) - \int_0^1 W_3^*(r)dW_4^*(r)}{\left\{\int_0^1 [W_3^*(r)]^2\,dr + \int_0^1 [W_4^*(r)]^2\,dr\right\}^{\frac{1}{2}}} \tag{60}$$

where $W_3^*(r) = [W_1(r) - W_3(r)]/\sqrt{2}$ and $W_4^*(r) = [W_2(r) - W_4(r)]/\sqrt{2}$ are independent standard Brownian motions. Note that the least squares bias $T\widehat{\pi}_3$ and $T\widehat{\pi}_4$ can also be obtained from (32) and (36).

HEGY suggest that π_3 and π_4 might be jointly tested, since they are both associated with the pair of nonstationary complex roots $\pm i$. Such joint testing might be accomplished by computing $F(\widehat{\pi}_3 \cap \widehat{\pi}_4)$ as for a standard F-test, although the distribution will not, of course, be the standard F-distribution. Due to the asymptotic independence of $\widehat{\pi}_3$ and $\widehat{\pi}_4$, Engle *et al.* (1993) show that the limiting distribution of $F(\widehat{\pi}_3 \cap \widehat{\pi}_4)$ is identical to that of $\frac{1}{2}[t^2{}_{\widehat{\pi}_3} + t^2{}_{\widehat{\pi}_4}]$, where the two individual components are given in (59) and (60). More details can be found in Smith and Taylor (1996) or Osborn and Rodrigues (1998).

Due to the asymptotic orthogonality of the regressors in (47), (54) and (58), these can be combined into the single test regression (44) without any effect on the asymptotic properties of the coefficient estimators.

3.4 Extensions to the HEGY Approach

Ghysels, Lee and Noh (1994), or GLN, consider further the asymptotic distribution of the HEGY test statistics for quarterly data and present some extensions. In particular, they propose the joint test statistics $F(\widehat{\pi}_1 \cap \widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4)$ and $F(\widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4)$, the former being an overall test of the null hypothesis $y_t \sim SI(1)$ and the latter a joint test of the seasonal unit roots implied by the summation operator $1 + L + L^2 + L^3$. Due to the two-sided nature of all F-tests, the alternative hypothesis in each case is that one or more of the unit root restrictions is not valid. Thus, in particular, these tests should not be interpreted as testing seasonal integration against stationarity for the process. From the asymptotic independence of $t_{\widehat{\pi}_i}$, $i = 1, ..., 4$, it follows that $F(\widehat{\pi}_1 \cap \widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4)$ has the same asymptotic distribution as $\frac{1}{4}\sum_{i=1}^{4}(t_{\widehat{\pi}_i})^2$. Thus, from (52), (56), (59) and (60), we have

$$F(\widehat{\pi}_1 \cap \widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4) \Rightarrow \frac{1}{4}\left\{ \frac{\left[\int_0^1 W_1^*(r)dW_1^*(r)\right]^2}{\int_0^1 [W_1^*(r)]^2 dr} + \frac{\left[\int_0^1 W_2^*(r)dW_2^*(r)\right]^2}{\int_0^1 [W_2^*(r)]^2 dr} + \frac{\left[\int_0^1 W_3^*(r)dW_3^*(r) + \int_0^1 W_4^*(r)dW_4^*(r)\right]^2}{\int_0^1 [W_3^*(r)]^2 dr + \int_0^1 [W_4^*(r)]^2 dr} + \frac{\left[\int_0^1 W_4^*(r)dW_3^*(r) - \int_0^1 W_3^*(r)dW_4^*(r)\right]^2}{\int_0^1 [W_3^*(r)]^2 dr + \int_0^1 [W_4^*(r)]^2 dr} \right\} \quad (61)$$

Hence, $F(\widehat{\pi}_1 \cap \widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4)$ is asymptotically distributed as the simple average of the squares of each of two Dickey-Fuller distributions, a DHF distribution with $\mathbb{S} = 2$ and (60). It is straightforward to see that a similar expression results for $F(\widehat{\pi}_2 \cap \widehat{\pi}_3 \cap \widehat{\pi}_4)$, which is a simple average of the squares of a Dickey-Fuller distribution, a DHF distribution with $\mathbb{S} = 2$ and (60).

GLN also observe that the usual test procedure of Dickey and Fuller [DF] (1979) can validly be applied in the presence of seasonal unit roots. However, this validity only applies if the regression contains sufficient augmentation. The essential reason derives from (39), so that the $SI(1)$ process $\Delta_{\mathbb{S}} y_y = \varepsilon_t$ can be written as

$$\Delta_1 y_t = \alpha_1 y_{t-1} + \phi_1 \Delta_1 y_{t-1} + ... + \phi_{\mathbb{S}-1} \Delta_1 y_{t-\mathbb{S}+1} + \varepsilon_t \qquad (62)$$

with $\alpha_1 = 0$ and $\phi_1 = ... = \phi_{\mathbb{S}-1} = -1$. With (62) applied as a unit root test regression, $t_{\widehat{\alpha}_1}$ asymptotically follows the usual DF distribution, as given in (52). See Ghysels, Lee and Siklos (1993), Ghysels, Lee and Noh (1994) and Rodrigues (1998b) for a more detailed discussion.

Beaulieu and Miron (1993) and Franses (1991) develop the HEGY approach for the case of monthly data.[2] This requires the construction of at least seven transformed variables, analogous to $y_{(1),t}$, $y_{(2),t}$ and $y_{(3),t}$ used in (41) to (43), and the estimation of twelve coefficients π_i $(i = 1, ..., 12)$. Beaulieu and Miron present the asymptotic distributions, noting that the t-type statistics corresponding to the two real roots of +1 and -1 each have the usual Dickey-Fuller form, while the remaining coefficients correspond to pairs of complex roots. In the Beaulieu and Miron parameterization, each of the five pairs of complex roots leads to a $t_{\widehat{\pi}_i}$ with a DHF distribution (again with $\mathbb{S} = 2$) and a $t_{\widehat{\pi}_i}$ with the distribution (60).

Although both Beaulieu and Miron (1993) and Franses (1991) discuss the use of joint F-type statistics for the two coefficients corresponding to a pair of complex roots, neither considers the use of the F-tests as in Ghysels *et al.* (1994) to test the overall Δ_{12} filter or the eleven seasonal unit roots. Taylor (1998) supplies critical values for these overall joint tests in the monthly case.

3.5 Multiple Tests and Levels of Significance

It is notable that many tests of the seasonal unit root null hypothesis involve tests on multiple coefficients. In particular, for the application of the HEGY test (44), Hylleberg *et al.* (1990) recommend that one-sided tests of π_1 and π_2 should be applied, with (π_3, π_4) either tested sequentially or jointly. The rationale for applying one-sided tests for

[2]The reparameterization of the regressors proposed for monthly data by Beaulieu and Miron (1993) is typically preferred because, in contrast to that of Franses (1991), the constructed variables are asymptotically orthogonal.

π_1, π_2 and π_3 is that it permits a test against stationarity, which is not the case when a joint F-type test is applied. Thus, the null hypothesis is rejected against stationarity only if the null hypothesis is rejected for each of these three tests. Many applied researchers have followed HEGY's advice, apparently failing to recognise the implications of this strategy for the overall level of significance for the implied joint test of $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$.

Let us assume that separate tests are applied to π_1 and π_2, with a joint test applied to (π_3, π_4), with each of these three tests applied at the same level of significance, α. Conveniently, these tests are mutually independent, due to the asymptotic orthogonality of the regressors, as discussed in Section 3.3. Therefore, the overall probability of rejecting the $SI(1)$ null hypothesis when it is true is

$$(1-\alpha)^3 \approx 1 - 3\alpha$$

for α small; see, for example Gourieroux and Monfort (1995). Thus, with $\alpha = .05$, the implied level of significance for the overall test is $1 - .95^3 = .14$, or approximately three times that of each individual test. With monthly data, the issue is even more important. If separate tests are applied to π_1, π_2, and each of the pairs (π_i, π_{i+1}) $(i = 3, 5, 7, 9, 11)$, each of these at the level of significance α, then the implied overall test of the $SI(1)$ null hypothesis is

$$1 - (1-\alpha)^7 \approx 7\alpha.$$

In this case, $\alpha = .05$ implies an overall level of significance of .30 and $\alpha = .01$ an overall level of .07. If the overall level of α is desired, then a simple way to (approximately) achieve this would be to use a level of α/k for each individual test, where k is the number of independent tests applied. This preserves the overall level of significance as (approximately) α while taking advantage of the one-sided tests available through direct use of the t-statistics.

In conclusion, the impact of multiple tests must be borne in mind when applying seasonal unit root tests. To date, however, these issues have received relatively little attention in this literature.

4 Near Seasonal Integration

As noted in Section 3.1 for the DHF test, $\Pr[t_{\widehat{\alpha}_s} < 0] = \Pr[\chi^2(\mathbb{S}) < \mathbb{S}]$ seems to be converging to 1/2 as $\mathbb{S}$ increases. However, for the periodicities typically considered this probability always exceeds 1/2. This phenomenon indicates that a standard normal distribution may not be a satisfactory approximation when the characteristic root is close to 1 and the sample size is moderate, as Chan and Wei (1987) point out. It is also a well established fact that the power of unit root tests is quite poor when the parameter of interest is in the neighborhood of unity (see, for example, Evans and Savin (1981, 1984) and Perron (1989)). This suggests a distributional gap between the standard distribution typically assumed under stationarity and the function of Brownian motions obtained when the DGP is a random walk. To close this gap, a new class of models have been proposed, which allow the characteristic root of a process to be in the neighborhood of unity. This type of process is often called near integrated. Important work concerning near integration in a conventional AR(1) process includes Bobkoski (1983), Cavanagh (1985), Phillips (1987, 1988), Chan and Wei (1987), Chan (1988, 1989) and Nabeya and Perron (1994).

In the exposition of the preceding sections, it has been assumed that the DGP is a special case of

$$y_t = \phi_{\mathbb{S}} y_{t-\mathbb{S}} + \varepsilon_t \tag{63}$$

with $\phi_{\mathbb{S}} = 1$ and $y_{-\mathbb{S}+1} = ... = y_0 = 0$. In this section we generalize the results by considering a class of processes characterized by an autoregressive parameter $\phi_{\mathbb{S}}$ close to 1. Analogously to the conventional near integrated AR(1), a noncentrality parameter c can be considered such that

$$\phi_{\mathbb{S}} = e^{c/N} \simeq 1 + \frac{c}{N}. \tag{64}$$

This characterizes a near seasonally integrated process, which can be locally stationary ($c < 0$), locally explosive ($c > 0$) or a conventional seasonal random walk ($c = 0$). This type of near seasonally integrated processes has been considered by Chan (1988, 1989), Perron (1992) and Rodrigues (1998c).

Similarly to the seasonal random walk, when the DGP is given by (63) and (64), and assuming that the observations are available for exactly N ($N = T/\mathbb{S}$) complete years, then

$$S_{sn} = \sum_{j=0}^{n-1} e^{\frac{jc}{N}} \varepsilon_{s,n-j} = \sum_{j=1}^{n} e^{(n-j)\frac{c}{N}} \varepsilon_{s,j} \qquad s = 1, ..., \mathbb{S} \tag{65}$$

This indicates that each season represents a near integrated process with a common noncentrality parameter c across seasons.

One of the main features of a process like (63) with $\phi_{\mathbb{S}} = e^{c/N}$, is that the FCLT and the CMT imply that

$$\frac{1}{N^{1/2}} y_{sn} = \frac{1}{N^{1/2}} S_{sn} \Rightarrow \sigma^2 J_{sc}(r), \qquad s = 1, ..., \mathbb{S} \tag{66}$$

where S_{sn} is the PSP corresponding to season s and $J_{sc}(r)$ is a Ornstein-Uhlenbeck processes and not a Brownian motion as in the seasonal random walk case. Note that, as indicated by for example, Phillips (1987) or Perron (1992), this diffusion process is generated by the stochastic differential equation

$$dJ_{sc}(r) = cJ_{sc}(r)dr + dW_s(r) \tag{67}$$

so that

$$J_{sc}(r) = W_s(r) + c \int_0^1 e^{(r-v)c} W_s(v) dv \tag{68}$$

and $J_{sc}(0) = 0$.

4.1 Power Functions for the DHF Test

The normalized least squares bias obtained from a DHF test regression when the DGP is (63) can be given by

$$T(\widehat{\phi}_{\mathbb{S}} - \phi_{\mathbb{S}}) = \frac{T^{-1} \sum_{t=1}^{T} y_{t-\mathbb{S}} \varepsilon_t}{T^{-2} \sum_{t=1}^{T} y_{t-\mathbb{S}}^2} \tag{69}$$

which can be written as a function of $\mathbb{S}$ independent processes

$$T(\widehat{\phi}_{\mathbb{S}} - \phi_{\mathbb{S}}) = \frac{(\mathbb{S}N)^{-1} \sum_{s=1}^{\mathbb{S}} \sum_{n=1}^{N} S_{s,n-1}(S_{s,n} - S_{s,n-1})}{(\mathbb{S}N)^{-2} \sum_{s=1}^{\mathbb{S}} \sum_{n=1}^{N} S_{s,n-1}^2}. \tag{70}$$

Recognizing that these PSPs are now of different nature than those obtained for the seasonal random walk case, and applying the results given by Phillips (1987, p.539), it can be shown that,

$$(\mathbb{S}N)^{-1} \sum_{s=1}^{\mathbb{S}} \sum_{n=1}^{N} S_{s,n-1}(S_{s,n} - S_{s,n-1}) \Rightarrow \frac{\sigma^2}{\mathbb{S}} \sum_{s=1}^{\mathbb{S}} \int_0^1 J_{sc}(r) dW_s(r) \tag{71}$$

and

$$(\mathbb{S}N)^{-2}\sum_{s=1}^{\mathbb{S}}\sum_{n=1}^{N}S_{s,n-1}^{2}\Rightarrow\frac{\sigma^{2}}{\mathbb{S}^{2}}\sum_{s=1}^{\mathbb{S}}\int_{0}^{1}J_{sc}^{2}(r)dr \tag{72}$$

where $J_{sc}(r)$ and $W_s(r)$, $s=1,...,\mathbb{S}$ are independent standard Ornstein-Uhlenbeck processes and independent standard Brownian motions, respectively.

Consequently, substituting (71) and (72) into (70) yields,

$$\frac{T}{\mathbb{S}}(\widehat{\phi}_{\mathbb{S}}-\phi_{\mathbb{S}})=\frac{\sum\limits_{s=1}^{\mathbb{S}}\int_{0}^{1}J_{sc}(r)dW_{s}(r)}{\sum\limits_{s=1}^{\mathbb{S}}\int_{0}^{1}J_{sc}^{2}(r)dr}. \tag{73}$$

It is also easy to see that the respective t-statistic converges to

$$t_{(\widehat{\phi}_{\mathbb{S}}-\phi_{\mathbb{S}})}=\frac{\sum\limits_{s=1}^{\mathbb{S}}\int_{0}^{1}J_{sc}(r)dW_{s}(r)}{\left[\sum\limits_{s=1}^{\mathbb{S}}\int_{0}^{1}J_{sc}^{2}(r)dr\right]^{\frac{1}{2}}}. \tag{74}$$

The result in (74) is the asymptotic power function for the DHF t-test. It is straightforward to observe that the distribution in (20) is a particular case of (74) with $c=0$. A more detailed analysis appears in Chan (1988, 1989), Perron (1992) and Rodrigues (1998c).

4.2 Power Functions for the HEGY Test

The examination of the HEGY procedure in a near seasonally integrated framework is slightly more involved. As indicated by Rodrigues (1998c), $(1-(1+\frac{c}{N})L^4)$ can be approximated by,

$$\left[1-\left(1+\frac{c}{4N}+O\left(N^{-2}\right)\right)L\right]\left[1+\left(1+\frac{c}{4N}+O\left(N^{-2}\right)\right)L\right]\times \\ \left[1+\left(1+\frac{c}{2N}+O\left(N^{-2}\right)\right)L^{2}\right] \tag{75}$$

The results provided by Jeganathan (1991), together with the orthogonality of the regressors in the HEGY test regression, yield the distributions of the HEGY statistics in the context of a near seasonally integrated process. In a similar way to the seasonal random walk case, the limiting behavior of the HEGY test statistics can be obtained from the following models:

$$\begin{aligned} z_{(1),t} &= \phi_1 z_{(1),t-1} + \varepsilon_t && (76)\\ z_{(2),t} &= -\phi_2 z_{(2),t-1} + \varepsilon_t && (77)\\ z_{(3),t} &= \phi_4 z_{(3),t-1} + \phi_3 z_{(3),t-2} + \varepsilon_t && (78) \end{aligned}$$

where $\phi_1 = e^{\frac{c}{T}} \simeq (1+\frac{c}{T})$, $\phi_2 = e^{\frac{c}{T}} \simeq (1+\frac{c}{T})$, $\phi_3 = e^{\frac{c}{T}} \simeq (1+\frac{c}{2T})$ and $\phi_4 = 0$.

Rodrigues (1998c) establishes the following limit results for the HEGY test regression:

$$T(\widehat{\pi}_i - \pi_i) \Rightarrow \frac{\int_0^1 J_{(i)c}(r)dW_{(i)}(r)}{\int_0^1 \left[J_{(i)c}(r)\right]^2 dr}, \quad i = 1,2 \qquad (79)$$

$$T(\widehat{\pi}_3 - \pi_3) \Rightarrow \frac{2\left\{\int_0^1 J_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J_{(4)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 \left[J_{(3)c}(r)\right]^2 dr + \int_0^1 \left[J_{(4)c}(r)\right]^2 dr\right\}} \qquad (80)$$

$$T(\widehat{\pi}_4 - \pi_4) \Rightarrow \frac{2\left\{\int_0^1 J_{(3)c}(r)dR_{(4)}(r) - \int_0^1 J_{(4)c}(r)dR_{(3)}(r)\right\}}{\left\{\int_0^1 \left[J_{(3)c}(r)\right]^2 dr + \int_0^1 \left[J_{(4)c}(r)\right]^2 dr\right\}} \qquad (81)$$

and

$$t_{\widehat{\pi}_i} \Rightarrow \frac{\int_0^1 J^*_{(i)c}(r)dW^*_{(i)}(r)}{\left\{\int_0^1 \left[J^*_{(i)c}(r)\right]^2 dr\right\}^{\frac{1}{2}}}, \quad i = 1,2 \qquad (82)$$

$$t_{\widehat{\pi}_3} \Rightarrow \frac{\left\{\int_0^1 J^*_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J^*_{(4)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 \left[J^*_{(3)c}(r)\right]^2 dr + \int_0^1 \left[J^*_{(4)c}(r)\right]^2 dr\right\}^{\frac{1}{2}}} \qquad (83)$$

$$t_{\widehat{\pi}_4} \Rightarrow \frac{\left\{\int_0^1 J^*_{(3)c}(r)dR_{(4)}(r) - \int_0^1 J^*_{(4)c}(r)dR_{(3)}(r)\right\}}{\left\{\int_0^1 \left[J^*_{(3)c}(r)\right]^2 dr + \int_0^1 \left[J^*_{(4)c}(r)\right]^2 dr\right\}^{\frac{1}{2}}} \qquad (84)$$

where

$$J_{(1)c}(r) = W_{(1)}(r) + c\int_0^1 e^{(r-v)c} W_{(1)}(v)dv$$

$$
\begin{aligned}
J_{(2)c}(r) &= W_{(2)}(r) + c\int_0^1 e^{(r-v)c} W_{(2)}(v)dv \\
J_{(3)c}(r) &= R_{(3)}(r) + c\int_0^1 e^{(r-v)c} R_{(3)}(v)dv \\
J_{(4)c}(r) &= R_{(4)}(r) + c\int_0^1 e^{(r-v)c} R_{(4)}(v)dv \\
W_{(1)}(r) &= W_1(r) + W_2(r) + W_3(r) + W_4(r) \\
W_{(2)}(r) &= W_1(r) - W_2(r) + W_3(r) - W_4(r) \\
R_{(3)}(r) &= W_1(r) - W_3(r) \\
R_{(4)}(r) &= W_2(r) - W_4(r)
\end{aligned}
$$

and where $J^*_{(i)c}(r) = \frac{1}{2}J_{(i)c}(r)$, $i = 1, 2$ and $J^*_{(j)c}(r) = \frac{1}{\sqrt{2}}J_{(j)c}(r)$, $j = 3, 4$.

Note that, since the joint tests computed from the HEGY test regression are averages of squared t-statistics (as indicated in Section 3.4), the distributions for the F-type tests typically considered can easily be obtained from (82) to (84).

One important result also put forward by Rodrigues (1998c) is that the distributions in (82) to (84) are still valid when we allow different noncentrality parameters for each factor in (4.13).

5 Conclusion

We have considered only the simple seasonal random walk case, which was used to present the general properties of seasonally integrated processes. It should be noted, however, that the effect of nonzero initial values and drifts on the distributions of the seasonal unit root test statistics can easily be handled substituting the standard Brownian motions by demeaned or detrended independent Brownian motions.

Among other issues not considered are the implications of autocorrelation and mean shifts for unit root tests. The first is discussed in detail in Ghysels et al. (1994), Hylleberg (1995) and Rodrigues and Osborn (1997). It is known that strong MA components can distort the power of these procedures. To a certain extend, however, these distortions can be corrected by augmenting the test regression with lags of the dependent variable.

The negative impact of mean shifts on the unit root test procedures, was noted by Ghysels (1991). Recently, Smith and Otero (1997) and Franses and Vogelsang (1998) have shown, using artificial data, that the HEGY test is strongly affected by seasonal mean shifts. This lead

Franses and Vogelsang to adapt the HEGY test so as to allow for deterministic mean shifts (Smith and Otero also present relevant critical values for the HEGY procedure in this context).

6 References

Abeysinghe, T. (1991), "Inappropriate use of Seasonal Dummies in Regression", *Economic Letters*, 36, pp. 175-179.

Abeysinghe, T. (1994), "Deterministic Seasonal Models and Spurious Regressions", *Journal of Econometrics*, 61, pp. 259-272.

Ahtola, J. and Tiao, G.C. (1987), "Distributions of Least Squares Estimators of Autoregressive Parameters for a Process with Complex Roots on the Unit Circle", *Journal of Time Series Analysis*, Vol. 8, pp. 1-14.

Barsky, R.B. and Miron, J.A. (1989), "The Seasonal Cycle and the Business Cycle", *Journal of Political Economy*, 97, pp. 503-535.

Beaulieu, J.J. and Miron, J.A. (1991), "The Seasonal Cycle in U.S. Manufacturing", *Economics Letters*, 37, pp. 115-118.

Beaulieu, J.J. and Miron, J.A. (1993), "Seasonal Unit Roots in Aggregate U.S. Data", *Journal of Econometrics*, 55, pp. 305-328.

Bobkoski, M.J. (1983) "Hypothesis testing in Nonstationary Time Series", Ph.D. thesis, University of Wisconsin, Madison.

Boswijk, H.P. and Franses, P.H. (1996) "Unit Roots in Periodic Autoregressions", *Journal of Time Series Analysis* 17, 221-245.

Box, G.E.P. and Jenkins, G.M. (1970), "*Time Series Analysis: Forecasting and Control*", San Francisco: Holden-Day.

Cavanagh, C., (1986), "Roots Local to Unity", Manuscript, Department of Economics, Harvard University, Cambridge, MA.

Canova, F. and Hansen, B.E. (1995), "Are Seasonal Patterns Constant Over Time? A Test for Seasonal Stability", *Journal of Business and Economic Statistics*, 13, pp. 237-252.

Chan, N.H. (1988), "The Parameter Inference for Nearly Nonstationary Time Series", *Journal of the American Statistical Association*, Vol. 83, No. 403, pp. 857-862.

Chan, N.H. (1989), "On the Nearly Nonstationary Seasonal Time Series", *The Canadian Journal of Statistics*, Vol. 17, No. 3, pp. 279-284.

Chan, N.H. and Wei, C.Z. (1987), "Asymptotic Inference for Nearly Nonstationary AR(1) Processes", *The Annals of Statistics*, Vol. 15, No.

3, pp. 1050-1063.

CHAN, N.H. AND WEI, C.Z. (1988), "Limiting Distributions of Least Squares Estimates of Unstable Autoregressive Processes", *The Annals of Statistics*, Vol. 16, pp. 367-401.

DICKEY, D.A., HASZA, D.P. AND FULLER, W.A. (1984), "Testing for Unit Roots in Seasonal Time Series", *Journal of the American Statistical Association*, 79, pp.355-367.

DICKEY, D.A. AND FULLER, W.A. (1979), "Distribution of the Estimators for Autoregressive Time Series with a Unit Root", *Journal of the American Statistical Association*, 74, pp. 427-431.

ENGLE AND GRANGER (1987), "Cointegration and Error Correction: Representation, Estimation and Testing", *Econometrics* 55, 251-276.

ENGLE, R.F., GRANGER, C.W.J., HYLLEBERG, S. AND LEE, H.S. (1993), "Seasonal Cointegration: The Japanese Consumption Function", *Journal of Econometrics*, 55, pp. 275-298.

EVANS, G.B.A. AND SAVIN, N.E. (1981), "Testing for Unit Roots: 1", *Econometrica*, Vol. 49, pp. 753-779.

EVANS, G.B.A. AND SAVIN, N.E. (1984), "Testing for Unit Roots: 2", *Econometrica*, Vol. 52, pp. 1241-1269.

FRANSES, P.H. (1991), "Seasonality, Nonstationarity and the Forecasting of Monthly Time Series", *International Journal of Forecasting*, 7, pp. 199-208.

FRANSES, P.H. (1994), "A Multivariate Approach to Modelling Univariate Seasonal Time Series", *Journal of Econometrics*, 63, pp. 133-151.

FRANSES, H.P., HYLLEBERG, S. AND LEE, H.S. (1995), "Spurious Deterministic Seasonality", *Economics Letters*, 48, pp. 249-256.

FRANSES, P.H. AND VOGELSANG, T.J. (1995), "Testing for Seasonal Unit Roots in the Presence of Changing Seasonal Means", Report 9532/A, Erasmus University Rotterdam.

FULLER, W.A. (1976), "*Introduction to Statistical Time Series*", John Wiley, New York.

GHYSELS, E., (1991), "On Seasonal Asymmetries and Their Implications on Deterministic and Stochastic Models of Seasonality", Mimeo, C.R.D.E., Université de Montréal.

GHYSELS, E. (1994), "On the Economics and Econometrics of Seasonality", in C.A. Sims, editor, *Advances in Econometrics*, pp. 257-316. Cambridge University Press (Cambridge).

GHYSELS, E., LEE, H.S. AND SIKLOS, P.L. (1993), "On the (Mis)Specification of Seasonality and its Consequences: An Empirical Investigation with US Data", *Empirical Economics*, 18, pp.747-760.

GHYSELS, E., LEE, H.S. AND NOH, J. (1994), "Testing for Unit Roots in Seasonal Time Series: Some Theoretical Extensions and a Monte Carlo Investigation", *Journal of Econometrics*, 62, pp.415-442.

GOURIEROUX, C. AND MONFORT, A. (1995), "*Statistics and Econometric Models*", Vol. II, Cambridge University Press.

HAMILTON, J.D. (1994), "*Time Series Analysis*", Princeton University Press.

HYLLEBERG, S. (1994), "Modelling Seasonal Variation, in Nonstationary Time Series Analysis and Cointegration, ed. by C. Hargreaves, Oxford University Press, pp. 153-178.

HYLLEBERG, S. (1995), "Tests for Seasonal Unit Roots: General to Specific or Specific to General?", *Journal of Econometrics*, 69, pp.5-25.

HYLLEBERG, S., JØRGENSEN, C. & SØRENSEN, N. K. (1993), "Seasonality in Macroeconomic Time Series", *Empirical Economics*, 18, pp. 321-335.

HYLLEBERG, S., ENGLE, R.F., GRANGER, C.W.J., AND YOO, B.S. (1990), "Seasonal Integration and Cointegration", *Journal of Econometrics*, 44, pp.215-238.

JEGANATHAN, P. (1991), "On the Asymptotic Behaviour of Least-Squares Estimators in AR Time Series with Roots Near the Unit Circle", *Econometric Theory* 7, 269-306.

NABEYA, S. AND PERRON, P. (1994), "Local Asymptotic Distribution Related to the AR(1) Model with Dependent Errors", *Journal of Econometrics*, 62, pp. 229-264.

OSBORN, D.R. (1990), "A Survey of Seasonality in UK Macroeconomic Variables", *International Journal of Forecasting*, 6, pp.327-336.

OSBORN, D.R. (1993), "Discussion on "Seasonal Cointegration: The Japanese Consumption Function", *Journal of Econometrics*, 55, pp. 299-303.

Osborn, D.R., Chui, A.P.L., Smith, J.P. and Birchenhall, C.R. (1988), "Seasonality and the Order of Integration for Consumption", *Oxford Bulletin of Economics and Statistics*, 50, pp.361-377.

Osborn, D.R. and Rodrigues, P.M.M. (1998), "The asymptotic distributions of seasonal unit root tests: A unifying approach.", University of Manchester, School of Economic Studies Discussion Paper Series No. 9811.

Perron, P. (1989), "The Calculation of the Limiting Distribution of the Least-Squares Estimator in a Near-Integrated Model", *Econometric Theory*, 5, pp. 241-255.

Perron, P. (1992), "The Limiting Distribution of the Least-Squares Estimator in Nearly Integrated Seasonal Models", *The Canadian Journal of Statistics*, Vol. 20, No. 2, pp. 121-134.

Phillips, P.C.B. (1987), "Towards a Unified Asymptotic Theory for Autoregression", *Biometrica* 74, 535-547.

Phillips, P.C.B. (1988), "Regression Theory for Near-Integrated Time Series", *Econometrica,* Vol. 56, 1021-1043;Rodrigues, P.M.M. (1998a), "*Inference in Seasonal Nonstationary Processes*", unpublished Ph.D. thesis, School of Economic Studies, University of Manchester.

Rodrigues, P.M.M. (1998b), "A Note on the Application of the DF Test to Seasonal Data", mimeo, School of Economic Studies, University of Manchester.

Rodrigues, P.M.M. (1998c), "Near Seasonal Integration", mimeo, School of Economic Studies, University of Manchester.

Rodrigues, P.M.M. and Osborn, D.R. (1997), "Evidence on testing for unit roots in seasonal time series", University of Manchester, School of Economic Studies Discussion Paper Series No. 9728.

Smith, R.J. and Taylor, A.M.R. (1996), "Additional Critical Values and Asymptotic Representations for Seasonal Unit Root Tests", *Journal of Econometrics*, forthcoming.

Taylor, A.M.R. (1998), "Additional Critical Values and Asymptotic Representations for Monthly Seasonal Unit Root Tests", *Journal of Time Series Analysis*, Vol. 19, pp. 349-368.

7 Appendix

The coefficient of determination (R^2) obtained from a regression like (14) is often used as a measure of the strength of the seasonal pattern. For the seasonal random walk DGP,

$$R^2 = \frac{\sum_{t=1}^{T}(\widehat{y}_t - \overline{y})^2}{\sum_{t=1}^{T}(y_t - \overline{y})^2} = \frac{\sum_{t=1}^{T}\widehat{y}_t^2 - T\overline{y}^2}{\sum_{t=1}^{T} y_t^2 - T\overline{y}^2} \tag{85}$$

$$\Rightarrow \frac{\sum_{s=1}^{\mathbb{S}}\left(\int_0^1 W_s(r)dr\right)^2 - \frac{1}{\mathbb{S}}\left[\int_0^1\left(\sum_{s=1}^{\mathbb{S}} W_s(r)\right)dr\right]^2}{\sum_{s=1}^{\mathbb{S}}\int_0^1 W_s^2(r)dr - \frac{1}{\mathbb{S}}\left[\int_0^1\left(\sum_{s=1}^{\mathbb{S}} W_s(r)\right)dr\right]^2}$$

Proof: It is a standard result for any regression containing an intercept (or a full set of seasonal dummy variables) that

$$\sum_{t=1}^{T}(\widehat{y}_t - \overline{y})^2 = \sum_{t=1}^{T}\widehat{y}_t^2 - T\overline{y}^2. \tag{86}$$

Similar arguments apply for the denominator of (86). Now,

$$\begin{aligned}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{y}_t^2 &= \frac{1}{(\mathbb{S}N)^2}\left[N\sum_{s=1}^{\mathbb{S}}\left(\frac{1}{N}\sum_{n=1}^{N} y_{sn}\right)^2\right] \\ &= \frac{1}{\mathbb{S}^2}\sum_{s=1}^{\mathbb{S}}\left(\frac{1}{N^{\frac{3}{2}}}\sum_{n=1}^{N} y_{sn}\right)^2 \\ &\Rightarrow \frac{\sigma^2}{\mathbb{S}^2}\sum_{s=1}^{\mathbb{S}}\left(\int_0^1 W_s(r)dr\right)^2.\end{aligned} \tag{87}$$

The convergence result for $\frac{1}{T^2}\sum_{t=1}^{T} y_t^2$ follows directly from part *d*) of Lemma 2.1, so that

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \Rightarrow \frac{\sigma^2}{\mathbb{S}^2}\sum_{s=1}^{\mathbb{S}}\int_0^1 W_s^2(r)dr. \tag{88}$$

The final result required is for $T^{-1}\overline{y}^2$.

$$
\begin{aligned}
T^{-1}\overline{y}^2 &= T^{-1}\left(\frac{\sum_{t=1}^{T} y_t}{T}\right)^2 = T^{-1}\left(\frac{\sum_{s=1}^{\mathbb{S}}\sum_{n=1}^{N} y_{sn}}{\mathbb{S}N}\right)^2 \\
&= \frac{1}{\mathbb{S}^3}\left(\frac{\sum_{s=1}^{\mathbb{S}}\sum_{n=1}^{N} y_{sn}}{N^{\frac{3}{2}}}\right)^2 \\
&\Rightarrow \frac{\sigma^2}{\mathbb{S}^3}\left[\int_0^1\left(\sum_{s=1}^{\mathbb{S}} W_s(r)\right)dr\right]^2 \qquad (89)
\end{aligned}
$$

Equations (87), (88) and (89) together yield (86).
■

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