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ARMA Representation of Two-Factor Models*

Nour Meddahi[†]

Résumé / *Abstract*

Beaucoup de modèles financiers sont spécifiés à travers des représentations structurelles. Néanmoins, la connaissance de formes réduites ARMA peut être utile pour l'analyse de fonction de réponses, le filtrage, la prévision, et pour les méthodes d'inférence statistique. Cette représentation ARMA est la forme analytique de l'état stable de la variable inobservable et est donc une alternative aux méthodes basées sur le filtre de Kalman. Dans cet article, nous dérivons les formules analytiques des racines moyenne-mobile d'un modèle à deux facteurs. Ensuite, nous proposons une application financière. Plus précisément, nous caractérisons la représentation GARCH(2,2) faible d'un modèle en temps continu et à volatilité stochastique quand la variance instantanée est la combinaison linéaire de deux processus auto-régressifs, comme pour les modèles affines, diffusion GARCH, CEV, Ornstein-Uhlenbeck et positifs, à fonctions propres, et SR-SARV.

Many financial time series models are specified through a structural representation. Nonetheless, knowing their reduced ARMA form may be useful for impulse response analysis, filtering, forecasting, and for purposes of statistical inference. This ARMA representation is the analytical steady-state of the unobservable variable and is therefore an alternative approach to Kalman filter-based methods. In this paper, we analytically derive the moving-average roots of a two-factor model. We then provide a financial application. More precisely, we characterize the weak GARCH(2,2) representation of continuous time stochastic volatility models when the variance process is a linear combination of two autoregressive processes, as in affine, GARCH diffusion, CEV, positive Ornstein-Uhlenbeck, eigenfunction, and SR-SARV processes.

Mots clés : modèles à deux facteurs, modèles structurels, représentation ARMA, racine moyenne-mobile, modèle en temps continu et à volatilité stochastique, représentation GARCH faible.

Keywords: Two-factor models, structural models, ARMA representation, Moving-average roots, continuous time stochastic volatility model, weak GARCH representation.

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1 Introduction

There are many examples of financial models where the variable of interest is defined as the sum of autoregressive factors of order one and (some times) a noise. Cases include interest rates in affine models (Vasicek, 1977; Cox, Ingersoll and Ross, 1985; Duffie and Kan, 1996; Dai and Singleton, 2000); the (stochastic) volatility of exchange rate returns (Taylor, 1986; Andersen, 1994; Harvey, Ruiz and Shephard, 1994; Jacquier, Polson and Rossi, 1994; Alizadeh, Brandt and Diebold, 2002; Barndorff-Nielsen and Shephard, 2002; Bollerslev and Zhou, 2002; Meddahi and Renault, 2002); the volatility of asset returns (Gallant, Hsu and Tauchen, 1999; Chernov, Gallant, Ghysels and Tauchen, 2002); the time between consecutive trades (Ghysels, Gourieroux and Jasiak, 2002). Accordingly, the time series properties of variables so defined is of particular interest. Following Granger and Morris (1976), when the number of autoregressive factors is p, the sum is an ARMA(p,p-1) if there is no noise and an ARMA(p,p) if there is. It is easy to characterize the autoregressive roots of the resulting ARMA process as they coincide with the factors. However, the characterization of the moving-average roots is less obvious. To the best of our knowledge, there is no analytic characterization of these roots when the order of the moving-average part is strictly higher than one. The main focus of the paper is to provide such a characterization. More precisely, we derive analytically the moving-average roots of ARMA(p,2) processes. The result, while not completely general, is of significant practical importance given that in most of the financial applications cited above the variable of interest is an ARMA(3,2) or an ARMA(2,2).

The exact characterization of the ARMA structure of a variable is important for impulse response analysis, filtering, forecasting, and for purposes of statistical inference. For example, the expected value of an unobserved variable, like volatility that is the expected value of squared returns, can be easily obtained from the ARMA representation of the observed variable. Typically, one finds a recursive formula for the expected value that looks like a GARCH equation (the filtering procedure from the ARMA representation). By using the results in Baillie and Bollerslev (1992), one obtains multi-period forecasts of the variable from its ARMA representation as well as forecasts of the future expected value of the observed variable by using the (GARCH-like) recursive formula of the expected value variable. The ARMA representation is also useful for statistical inference. For instance, under Gaussianity, one can compute the maximum-likelihood estimator. One can also use the ARMA representation for estimation purposes when the innovations of the ARMA are not martingale difference sequences, an approach that is adopted by Francq and Zakoïan (2000) who use the weak GARCH structure of returns to estimate the volatility parameters. It should also be noted that the (GARCH-like) recursive equation is the (analytical) steady-state of the Kalman filter. Hence, it can be used instead of the Kalman filter in a QML estimation procedure for volatility models.

The paper presents a financial application that is of particular relevance for volatility modeling. In the application, we characterize the GARCH(2,2) structure of a two-factor continuous time stochastic volatility model. More precisely, for a given frequency of observations, we characterize the ARMA(2,2) structure of the squared returns when the variance is a linear combination of two autoregressive processes: examples include affine processes (Heston, 1993; Duffie, Pan and Singleton, 2000); GARCH diffusion (Nelson, 1990); the CEV process (Meddahi and Renault, 2002); the positive Ornstein-Uhlenbeck processes (Barndorff-Nielsen and Shephard, 2001), the SR-SARV models (Andersen, 1994; Meddahi and Renault, 2002); and the eigenfunction stochastic volatility models (Meddahi, 2001). The ARMA representation of the squared returns is the weak GARCH structure of Drost and Nijman (1993). In Drost and Werker (1996), the weak GARCH(1,1) structure of the return process is characterized when the variance depends on one autoregressive process. We extend this result to the two-factor volatility model. This extension is important for practical purposes since it has been well established that two factors are needed for correctly modeling volatility: one factor to capture the persistence of the volatility and a second to deal with fat-tails (e.g., Engle and Lee, 1999; Meddahi, 2001; Bollerslev and Zhou, 2002; Gallant, Hsu and Tauchen, 1999; Chernov et al., 2002).

The remainder of the paper is organized as follows. Section 2 contains the basic results: we derive the roots of an MA(2) process when its first and second autocorrelations are known. Then we derive the ARMA(2,2) representation of a process defined as the sum of two autoregressive factors and a noise (two-factor model). In Section 3, we derive the weak GARCH(2,2) representation of continuous time stochastic volatility models when the variance depends linearly on two autoregressive factors. We then study the empirical implications of the weak GARCH representation. Section 4 concludes the paper. All proofs can be found in the Appendix.

2 Moving-average roots of two-factor models

2.1 An overview

Consider a random variable of interest denoted by y_t and defined as the sum of two AR(1) processes plus noise (and called two-factor model), i.e.,

$$y_t = f_{1,t-1} + f_{2,t-1} + u_t, \text{ with} \tag{2.1}$$

$$f_{i,t} = \omega_i + \gamma_i f_{i,t-1} + v_{i,t}, \quad \mid \gamma_i \mid < 1, \quad \text{for } i = 1, 2, \tag{2.2}$$

where $(v_{1,t}, v_{2,t}, u_t)^\top$ is a white noise. In order to characterize the ARMA dynamics of y_t, it is of interest to define the variable z_t as

$$z_t \equiv (1 - \gamma_1 L)(1 - \gamma_2 L) y_t - (1 - \gamma_2)\omega_1 - (1 - \gamma_1)\omega_2. \tag{2.3}$$

Observe that

$$z_t = v_{1,t-1} - \gamma_2 v_{1,t-2} + v_{2,t-1} - \gamma_1 v_{2,t-2} + u_t - (\gamma_1 + \gamma_2)u_{t-1} + \gamma_1\gamma_2 u_{t-2}. \tag{2.4}$$

Hence, given that $(v_{1,t}, v_{2,t}, u_t)^\top$ is a white noise, we have that (see the Appendix for a proof)

$$\forall h, \mid h \mid > 2, \quad Cov(z_t, z_{t-h}) = 0, \tag{2.5}$$

i.e., z_t is a moving-average process of order two. As a consequence, given the relationship (2.3), y_t is an ARMA(2,2).[1] In addition, the autoregressive coefficients of y_t are γ_1 and γ_2. Finally, the moving-average roots of y_t are those of z_t.[2]

Given that z_t is a MA(2) process, it admits the following representation:

$$z_t = (1 - \lambda_1 L)(1 - \lambda_2 L)\eta_t, \tag{2.6}$$

where η_t is a white noise, L the lag operator and λ_1 and λ_2 are complex numbers with modulus smaller than one, i.e. $\mid \lambda_i \mid < 1$ for $i = 1, 2$. We can rewrite (2.6) as the following representation

$$z_t = \eta_t - \beta_1 \eta_{t-1} - \beta_2 \eta_{t-2}, \quad \text{where} \tag{2.7}$$

$$\beta_1 = \lambda_1 + \lambda_2 \text{ and } \beta_2 = -\lambda_1 \lambda_2. \tag{2.8}$$

It is worth noting that without further assumption, the process η_t is not a martingale difference sequence (m.d.s.). To the best of our knowledge, the conditional normality and homoskedasticity of $(v_{1,t}, v_{2,t}, u_t)^\top$ is the unique example where η_t is m.d.s. and, indeed, normal and i.i.d. (independent and identically distributed); see Meddahi and Renault (2002). These authors also show that this m.d.s. property does not hold for volatility models given that $(v_{1,t}, v_{2,t}, u_t)^\top$ is heteroskedastic, as in the application considered in the subsequent section. However, Meddahi and Renault (2002) show that the process η_t is more restricted than a white noise given that the following condition holds

$$E[\eta_t - \beta_1 \eta_{t-1} - \beta_2 \eta_{t-2} \mid \eta_{t-3}] = 0.$$

Such multi-period conditional moment restrictions were introduced by Hansen (1985) and studied in detail by Hansen, Heaton and Ogaki (1987), Hansen and Singleton (1986), West (2001) and Kuersteiner (2002).

By combining (2.3) and (2.7), one easily obtains the ARMA(2,2) representation of y_t:

$$y_t = (1 - \gamma_2)\omega_1 + (1 - \gamma_1)\omega_2 + (\gamma_1 + \gamma_2) y_{t-1} - \gamma_1 \gamma_2 y_{t-2} + \eta_t - \beta_1 \eta_{t-1} - \beta_2 \eta_{t-2}. \tag{2.9}$$

[1]When $\gamma_1 = \gamma_2$, it is easy to show by a similar proof that y_t is indeed an ARMA(1,1).

[2]However, for some specific values of the model parameters, one moving-average root of y_t coincides with one autoregressive roots of y_t (i.e., γ_1^{-1} or γ_2^{-1}) and y_t becomes an ARMA(1,1).

Let us now define $m_{t-1}[y]$ as the best linear predictor of y_t on the Hilbert-space generated by $\{1, y_\tau, \tau \leq t-1\}$. Then we naturally obtain

$$m_t[y] = \omega + \alpha_1 y_t + \alpha_2 y_{t-1} + \beta_1 m_{t-1}[y] + \beta_2 m_{t-2}[y], \tag{2.10}$$

where

$$\omega = (1-\gamma_2)\omega_1 + (1-\gamma_1)\omega_2,\ \ \alpha_1 = \gamma_1 + \gamma_2 - \beta_1, \alpha_2 = -\gamma_1\gamma_2 - \beta_2.$$

The main interesting feature of (2.10) is that it looks like a GARCH(2,2) equation (Bollerslev, 1986). In fact, this recursive equation is useful in both estimation purposes (as in Francq and Zakoïan, 2000) and forecasting procedures (as in Baillie and Bollerslev, 1992). An additional advantage of (2.10) is that it corresponds to the steady-state of the Kalman filter of the variable $(f_{1,t}+f_{2,t})$. Hence, one can use (2.10) instead of a Kalman filter in a QML estimation procedure, as in Harvey, Ruiz, Shephard (1994) for log-normal volatility models and Barndorff-Nielsen and Shephard (2002) for realized volatility models.

2.2 Roots of a MA(2) process

We will now characterize the moving-average roots of the process z_t in terms of its first and second autocorrelations. For this purpose, we use the following notation: $sign(a) = 1$ if $a > 0$ and $sign(a) = -1$ if $a < 0$.

Proposition 2.1 Roots of a MA(2) process. *Let z_t be a moving-average process of order 2, MA(2), with a zero mean, where ρ_1 and ρ_2 are its first and second autocorrelation; i.e.,*

$$\rho_1 \equiv \frac{Cov[z_t, z_{t-1}]}{Var[z_t]} \text{ and } \rho_2 \equiv \frac{Cov[z_t, z_{t-2}]}{Var[z_t]}, \tag{2.11}$$

and we assume that ρ_2 is non-zero. Then we have:
1) the process z_t admits the representation (2.7) where β_1 and $Var[\eta_t]$ are given by

$$\beta_1 = \frac{\beta_2}{1-\beta_2}\frac{\rho_1}{\rho_2} \text{ and } \ Var[\eta_t] = \frac{Var[z_t]}{1+\beta_1^2+\beta_2^2}, \tag{2.12}$$

while β_2 is given by:

$$\begin{aligned} &\text{if } \rho_1 = 0 : \beta_2 = \frac{1}{2}\left(-{\rho_2}^{-1} - sign(\rho_2)\sqrt{\rho_2^{-2} - 4}\right), \\ &\text{if } \rho_1 \neq 0 : \beta_2 = \frac{2u+1-\sqrt{4u+1}}{2u} \text{, where} \end{aligned} \tag{2.13}$$

$$u = 2^{-1}\rho_2^2\rho_1^{-2}\left[-2 - \rho_2^{-1} + sign(\rho_2)\sqrt{(2+{\rho_2}^{-1})^2 - 4\rho_1^2\rho_2^{-2}}\right]; \tag{2.14}$$

2) besides, the process z_t admits the representation (2.6) where λ_1 and λ_2 are given by

$$\lambda_1 = \frac{\beta_1 - \sqrt{\beta_1^2+4\beta_2}}{2} \ \text{ and } \ \lambda_2 = \frac{\beta_1 + \sqrt{\beta_1^2+4\beta_2}}{2}. \tag{2.15}$$

In the rest of the section, we consider the characterization of the moving-average roots of the two-factor model defined previously. This example is important in the financial literature and, indeed, the application we consider later is a special case of it. It is worth noting that in characterizing the moving-average roots of a general ARMA(p,2) model, one has to adopt the same approach as we did above, i.e., characterize the moving-average part and its properties (such as variance, first and second autocorrelations) and use the previous proposition. In particular, this approach can be adopted if one is interested in the ARMA representation of a variable defined as the sum of the components of an exact discretization of trivariate Ornstein-Uhlenbeck process (see Bergström, 1984, 1990; Comte and Renault, 1996). We adopt the same approach for the leading example we consider.

2.3 ARMA(2,2) representation of two-factor models

The ARMA(2,2) representation of the process y_t defined in (2.1) is given in (2.9). In order to complete this representation, we will use the previous proposition to compute the moving average parameters of the process z_t. Therefore, the characterization will be achieved when we get the variance of z_t and its first and second autocorrelations.

We will derive these characteristics of z_t in two different manners. The first one derive them in terms of the structural parameters, i.e., the autoregressive coefficients, γ_1 and γ_2, and variance-covariance matrix of $(v_{1,t}, v_{2,t}, u_t)^\top$. We will adopt this structural characterization in the weak GARCH application in Section 3. For this purpose, for any real numbers $\gamma_1, \gamma_2, V_{11}, V_{22}, V_{33}, V_{31}, V_{32}$, we introduce the notation:

$$\begin{aligned} G_{2,0}(\gamma_1, \gamma_2, V_{11}, V_{22}, V_{33}, V_{31}, V_{32}) \equiv & (1+\gamma_2^2)V_{11} + (1+\gamma_1^2)V_{22} + (1+(\gamma_1+\gamma_2)^2 + (\gamma_1\gamma_2)^2)V_{33} \\ & - 2(\gamma_1+\gamma_2+\gamma_1\gamma_2^2)V_{31} - 2(\gamma_1+\gamma_2+\gamma_1^2\gamma_2)V_{32} \end{aligned} \tag{2.16}$$

$$\begin{aligned} G_{2,1}(\gamma_1, \gamma_2, V_{11}, V_{22}, V_{33}, V_{31}, V_{32}) \equiv & -\gamma_2 V_{11} - \gamma_1 V_{22} - (\gamma_1+\gamma_2)(1+\gamma_1\gamma_2)V_{33} \\ & + (1+2\gamma_1\gamma_2+\gamma_2^2)V_{31} + (1+2\gamma_1\gamma_2+\gamma_1^2)V_{32}, \end{aligned} \tag{2.17}$$

$$G_{2,2}(\gamma_1, \gamma_2, V_{33}, V_{31}, V_{32}) \equiv \gamma_1\gamma_2 V_{33} - \gamma_2 V_{31} - \gamma_1 V_{32}. \tag{2.18}$$

Proposition 2.2 Characteristics of z_t from the structural representation of y_t. *Let y_t, $f_{1,t}$, $f_{2,t}$ and z_t be the processes defined in (2.1), (2.2) and (2.3). Then:*

$$\begin{aligned} Var[z_t] = & G_{2,0}(\gamma_1, \gamma_2, Var[v_{1,t}], Var[v_{2,t}], Var[u_t], Cov[u_t, v_{1,t}], Cov[u_t, v_{2,t}]) \\ & + 2(1+\gamma_1\gamma_2)Cov[v_{1,t}, v_{2,t}], \end{aligned} \tag{2.19}$$

$$\begin{aligned} Cov[z_t, z_{t-1}] = & G_{2,1}(\gamma_1, \gamma_2, Var[v_{1,t}], Var[v_{2,t}], Var[u_t], Cov[u_t, v_{1,t}], Cov[u_t, v_{2,t}]) \\ & - (\gamma_1+\gamma_2)Cov[v_{1,t}, v_{2,t}], \end{aligned} \tag{2.20}$$

$$Cov[z_t, z_{t-2}] = G_{2,2}(\gamma_1, \gamma_2, Var[u_t], Cov[u_t, v_{1,t}], Cov[u_t, v_{2,t}]), \tag{2.21}$$

where the functions $G_{2,0}(.)$, $G_{2,1}(.)$ and $G_{2,2}(.)$ are defined in (2.16), (2.17) and (2.18).

In the second approach, we derive the characteristics of z_t in terms of the autoregressive coefficients, the variance of y_t, and the first and second autocovariances of y_t.

Proposition 2.3 Characteristics of z_t from the reduced form of y_t. *Let y_t, $f_{1,t}$, $f_{2,t}$ and z_t be the processes defined in (2.1), (2.2) and (2.3). Then:*

$$\begin{aligned} Var[z_t] = (1+\gamma_1^2\gamma_2^2+(\gamma_1+\gamma_2)^2)Var[y_t] - 2(\gamma_1+\gamma_2)(1+\gamma_1\gamma_2)Cov[y_t, y_{t-1}] \\ + 2\gamma_1\gamma_2 Cov[y_t, y_{t-2}], \end{aligned} \tag{2.22}$$

$$\begin{aligned} Cov[z_t, z_{t-1}] = -(1+\gamma_1\gamma_2)(\gamma_1+\gamma_2)Var[y_t] + (1+(\gamma_1+\gamma_2)^2+\gamma_1\gamma_2)Cov[y_t, y_{t-1}] \\ - (\gamma_1+\gamma_2)Cov[y_t, y_{t-2}], \end{aligned} \tag{2.23}$$

$$Cov[z_t, z_{t-2}] = \gamma_1\gamma_2 Var[y_t] - (\gamma_1+\gamma_2)Cov[y_t, y_{t-1}] + Cov[y_t, y_{t-2}]. \tag{2.24}$$

3 Continuous time weak GARCH(2,2)

3.1 Weak GARCH(2,2) representation of two-factor volatility models

In this section, we characterize the weak GARCH representation of models where the variance is a linear combination of two autoregressive processes, as affine processes (Heston, 1993; Duffie, Pan and Singleton, 2000), GARCH diffusion (Nelson, 1990), CEV process (Meddahi and Renault, 2002), positive Ornstein-Uhlenbeck processes (Barndorff-Nielsen and Shephard, 2001), SR-SARV models (Andersen, 1994; Meddahi and Renault, 2002) and the eigenfunction stochastic volatility models (Meddahi, 2001).[3] Such a characterization is important because it gives the relationship between the parameters of the continuous time model and those of the discrete time model. This characterization is considered by Drost and Werker (1996) when the variance process depends linearly on one factor and is used by several authors, in particular Andersen and Bollerslev (1998), Drost, Nijman and Werker (1998), Andersen, Bollerslev and Lange (1999), Andreou and Ghysels (2002). However, it is now well established that a one factor stochastic volatility model does not describe well the data. Typically, one needs two factors, one to capture the persistence of the volatility and a second to capture the tails of the returns; see, for instance, Meddahi (2001), Chernov et al. (2002). This approach is in line with the GARCH model of Engle and Lee (1999).[4] In conclusion, it is of interest to characterize the weak GARCH representation of two-factor models.

[3]By using the results of the previous section, one can also easily derive the weak GARCH(2,2) representation of a temporal aggregation of a discrete time weak GARCH(2,2) or a cross-sectional aggregation of two independent GARCH(1,1) processes as in Nijman and Sentana (1996).

[4]An alternative approach to the two-factor model is to consider a one-factor model with jumps as in Andersen, Benzoni and Lund (2002) and Pan (2002); for a comprehensive empirical comparison of these approaches, see Chernov et al. (2002).

In the sequel, we assume that the log-price of an asset or the log of an exchange rate, denoted by x_t, is a continuous time model given by

$$dx_t = \sigma_t dW_t, \tag{3.1}$$

with σ_t^2 a linear combination of two autoregressive processes, i.e.,

$$\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2, \quad d\sigma_{i,t}^2 = k_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_i \sigma_{i,t}^{\delta_i} dW_{i,t}, \text{ for } i = 1, 2, \ 1 \leq \delta_i \tag{3.2}$$

where W_t, $W_{1,t}$, and $W_{2,t}$ are three independent standard Brownian motion processes. Note that we can allow for dependence between $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$; we only need that these two processes be uncorrelated. An example of autoregressive processes that are uncorrelated but dependent can be obtained by considering two eigenfunctions (associated with different eigenvalues) of the infinitesimal generator of the state variable (Hansen and Scheinkman, 1995). This approach is adopted by Meddahi (2001) where the Eigenfunction Stochastic Volatility (ESV) models are introduced.[5] However, we maintain the assumption that W_t is independent with $W_{1,t}$, and $W_{2,t}$ and, hence, we exclude the leverage effect, which is a usual assumption in the weak GARCH setting. This model is a special case of SR-SARV(2) models of Meddahi and Renault (2002) and is, hence, closed under temporal aggregation.[6] Another example of SR-SARV is the positive Ornstein-Uhlenbeck of Barndorff-Nielsen and Shephard (2001). The following weak GARCH(2,2) characterization holds for any SR-SARV(2) model. However, we will focus on the model (3.2) since this is the most popular model in the empirical literature, but a similar proof may be easily developed.

The general formulation of $\sigma_{i,t}^2$ is called a CEV process. When $\delta_i = 1$, we have the usual square-root process (Feller, 1951; Cox, Ingersoll and Ross, 1984) and we have the GARCH diffusion process when $\delta_i = 2$ (Wong, 1964; Nelson, 1990). We will assume that the second moment of $\sigma_{i,t}^2$ is finite in order to ensure the existence of the fourth moment of the returns, an assumption needed in the weak GARCH formulation. This assumption holds always when $\delta_i < 2$ and never when $\delta_i > 2$.[7] When $\delta_i = 2$, this assumption is ensured when $\sigma_i^2 < 2k_i$. Thus, we assume that $\delta_i \leq 2$, and $\sigma_i^2 < 2k_i$ when $\delta_i = 2$.
For a given h, $h > 0$, we denote by $\varepsilon_{th}^{(h)}$ the (log) return of x_t over the period $[(t-1)h, th]$:

$$\varepsilon_{th}^{(h)} = x_{th} - x_{(t-1)h} = \int_{(t-1)h}^{th} \sigma_u dW_u. \tag{3.3}$$

Meddahi and Renault (2002) showed that the discrete time process $\{\varepsilon_{th}^{(h)}, t \in \mathbf{N}\}$ is a SR-SARV(2). In addition, given that we have ruled out the leverage effect, the results of Meddahi

[5]For an alternative approach using eigenfunction (or principal components), see Chen, Hansen and Scheinkman (2000). For a review of the properties of the infinitesimal generator operator of a diffusion, see Aït-Sahalia, Hansen and Scheinkman (2001).

[6]Meddahi and Renault (2002) show that an exact discretization of a continuous time SR-SARV(p) model is a discrete time SR-SARV(p) model of Andersen (1994).

[7]Therefore, we exclude the example of Jones (2002) who considers a CEV volatility process with $\delta_i > 2$.

and Renault (2002) imply that $\{\varepsilon_{th}^{(h)}, t \in \mathbf{N}\}$ is a weak GARCH(2,2) of Drost and Nijman (1993). We now characterize the coefficient of the weak GARCH(2,2) representation. For this purpose, we will first derive the structural representation of the squared returns such as (2.1) and (2.2).[8] Then we will get the weak GARCH representation by using both Proposition 2.1 and Proposition 2.2.

Proposition 3.1 Structural representation of squared returns. *Let $\{x_t, t \in I\!R^+\}$ be the continuous time SV model defined by (3.1) and (3.2). For any real h, $h > 0$, define the returns $\{\varepsilon_{th}^{(h)}, t \in \mathbf{N}\}$ by (3.3). Then we have:*

$$(\varepsilon_{th}^{(h)})^2 = f_{1,(t-1)h}^{(h)} + f_{2,(t-1)h}^{(h)} + u_{th}^{(h)}, \text{ with} \tag{3.4}$$

$$f_{i,(t-1)h}^{(h)} = \theta_i h + \frac{1-\exp[-k_i h]}{k_i}(\sigma_{i,(t-1)h}^2 - \theta_i), \text{ for } i = 1,2, \quad \text{and} \tag{3.5}$$

$$\begin{aligned} u_{th}^{(h)} = & \sum_{i=1}^{2} \int_{(t-1)h}^{th} \exp[-k_i(u-(t-1)h)] \left[\int_{(t-1)h}^{u} \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}\right] du \\ & + 2\int_{(t-1)h}^{th} (\int_{(t-1)h}^{u} \sigma_s dW_s)\sigma_u dW_u. \end{aligned} \tag{3.6}$$

In addition, we have

$$f_{i,th}^{(h)} = \omega_i^{(h)} + \gamma_i^{(h)} f_{i,(t-1)h}^{(h)} + v_{i,th}^{(h)}, \text{ with} \tag{3.7}$$

$$\omega_i^{(h)} = \theta_i h(1-\exp[-k_i h]), \ \gamma_i^{(h)} = \exp[-k_i h], \tag{3.8}$$

$$v_{i,th}^{(h)} = \frac{1-\exp[-k_i h]}{k_i} \exp[-k_i h] \int_{(t-1)h}^{th} \exp[k_i(u-(t-1)h)]\sigma_i \sigma_{i,u}^{\delta_i} dW_{i,u}. \tag{3.9}$$

Finally, the process $(v_{1,th}^{(h)}, v_{2,th}^{(h)}, u_{th}^{(h)})^\top$ is a white noise and we have

$$Var[v_{i,th}^{(h)}] = \frac{(1-\exp[-k_i h])^2(1-\exp[-2k_i h])}{2k_i^3}\sigma_i^2 E[\sigma_{i,t}^{2\delta_i}], \tag{3.10}$$

$$\begin{aligned} Var[u_{th}^{(h)}] = & \sum_{i=1}^{2} \frac{4\exp[-k_i h] - \exp[-2k_i h] + 2k_i h - 3}{2k_i^3} \ \sigma_i^2 E[\sigma_{i,t}^{2\delta_i}] \\ & + 4\sum_{i=1}^{2} \frac{\exp[-k_i h] - 1 + k_i h}{k_i^2} \ Var[\sigma_{i,t}^2] + 2h^2 \ (\theta_1+\theta_2)^2, \end{aligned} \tag{3.11}$$

$$Cov[v_{1,th}^{(h)}, v_{2,th}^{(h)}] = 0, \quad \text{and} \quad Cov[u_{th}^{(h)}, v_{i,th}^{(h)}] = \frac{(1-\exp[-k_i h])^3}{2k_i^3}\sigma_i^2 E[\sigma_{i,t}^{2\delta_i}]. \tag{3.12}$$

[8]This structural representation is exactly the SR-SARV representation of Meddahi and Renault (2002).

This proposition can be summarized as follows. The equation (3.4) means that the squared return $(\varepsilon_{th}^{(h)})^2$ is a linear combination of the two processes $f_{1,(t-1)h}^{(h)}$ and $f_{2,(t-1)h}^{(h)}$, plus the noise $u_{th}^{(h)}$. Each process $f_{i,(t-1)h}^{(h)}$, $i=1,2$, is an affine function of $\sigma_{i,(t-1)h}^2$ (see (3.5)) and, hence, is an AR(1) process with the innovation process $v_{i,(t-1)h}^{(h)}$ (see (3.7)). By using the structure of the noise $u_{th}^{(h)}$ and the innovations $v_{i,th}^{(h)}$ given respectively in (3.6) and (3.9), we characterize their variances in (3.11) and (3.10) respectively. Finally, given that $v_{i,th}^{(h)}$ depends only on the process $W_{i,t}$ and given the independence of $W_{1,t}$ and $W_{2,t}$, the processes $v_{1,th}^{(h)}$ and $v_{2,th}^{(h)}$ are independent (and uncorrelated). However, $u_{th}^{(h)}$ depends on both $W_{1,t}$ and $W_{2,t}$ and is therefore correlated with $v_{1,th}^{(h)}$ and $v_{2,th}^{(h)}$ (see (3.12)).

In Proposition 3.1, we need $E[\sigma_{i,t}^{2\delta_i}]$ and $Var[\sigma_{i,t}^2]$ which are given (see Meddahi, 2001)

$$E[\sigma_{i,t}^{2\delta_i}] = \theta_i,\ Var[\sigma_{i,t}^2] = \frac{\theta_i\sigma_i^2}{2k_i} \text{ if } \ \delta_i = 1, \text{ and } \ E[\sigma_{i,t}^{2\delta_i}] = \frac{2\theta_i^2 k_i}{2k_i - \sigma_i^2},\ Var[\sigma_{i,t}^2] = \frac{\theta_i^2\sigma_i^2}{2k_i - \sigma_i^2} \text{ if } \delta_i = 2.$$

In order to characterize the weak GARCH(2,2) representation of the returns $(\varepsilon_{th}^{(h)})$, we proceed as in the previous section. Hence, let $z_{th}^{(h)}$ be the process defined by

$$z_{th}^{(h)} = (\varepsilon_{th}^{(h)})^2 - (\gamma_1^{(h)} + \gamma_2^{(h)})(\varepsilon_{(t-1)h}^{(h)})^2 + \gamma_1^{(h)}\gamma_2^{(h)}(\varepsilon_{(t-2)h}^{(h)})^2 - \omega^{(h)},$$

where $\gamma_i^{(h)}$, i=1,2, are given in (3.8) and $\omega^{(h)}$ is defined by

$$\omega^{(h)} = (1-\gamma_1^{(h)})(1-\gamma_2^{(h)})h(\theta_1+\theta_2). \tag{3.13}$$

Now we have to compute the first and second autocorrelation of $z_{th}^{(h)}$ denoted by $\rho_1^{(h)}$ and $\rho_2^{(h)}$. By combining the results of Proposition 2.2 and Proposition 3.1, it is clear that

$$\rho_1^{(h)} = \frac{Cov[z_{th}^{(h)}, z_{(t-1)h}^{(h)}]}{Var[z_{th}^{(h)}]} \text{ and } \rho_2^{(h)} = \frac{Cov[z_{th}^{(h)}, z_{(t-2)h}^{(h)}]}{Var[z_{th}^{(h)}]}, \text{ with} \tag{3.14}$$

$$\begin{aligned}
Var[z_{th}^{(h)}] &= G_{2,0}(\gamma_1^{(h)}, \gamma_2^{(h)}, Var[v_{1,th}^{(h)}], Var[v_{2,th}^{(h)}], Var[u_{th}^{(h)}], Cov[u_{th}^{(h)}, v_{1,th}^{(h)}], Cov[u_{th}^{(h)}, v_{2,th}^{(h)}]) \\
Cov[z_{th}^{(h)}, z_{(t-1)h}^{(h)}] &= G_{2,1}(\gamma_1^{(h)}, \gamma_2^{(h)}, Var[v_{1,th}^{(h)}], Var[v_{2,th}^{(h)}], Var[u_{th}^{(h)}], Cov[u_{th}^{(h)}, v_{1,th}^{(h)}], Cov[u_{th}^{(h)}, v_{2,th}^{(h)}]) \\
Cov[z_{th}^{(h)}, z_{(t-2)h}^{(h)}] &= G_{2,1}(\gamma_1^{(h)}, \gamma_2^{(h)}, Var[u_{th}^{(h)}], Cov[u_{th}^{(h)}, v_{1,th}^{(h)}], Cov[u_{th}^{(h)}, v_{2,th}^{(h)}])
\end{aligned}$$

where $\gamma_i^{(h)}$, $Var[v_{i,th}^{(h)}]$, $Var[u_{th}^{(h)}]$ and $Cov[u_{th}^{(h)}, v_{i,th}^{(h)}]$, for $i=1,2$, are given respectively in (3.8), (3.10), (3.11) and (3.12). The last step in characterizing the weak GARCH(2,2) representation is the computation of the MA roots by using Proposition 2.1. For this purpose, we need to know if $\rho_1^{(h)}$ equals zero or not and the sign of $\rho_2^{(h)}$. The coefficient $\rho_1^{(h)}$ is generically non-zero. However, it is not possible to give the sign of $\rho_2^{(h)}$ without specifying all the parameters. For instance, in the empirical illustration we consider later, $\rho_2^{(h)}$ is positive when one considers intra-daily returns but becomes positive with weekly returns.

We can now characterize the weak GARCH(2,2) representation of the returns $\varepsilon_{th}^{(h)}$:

Proposition 3.2 Weak GARCH(2,2) representation of returns. *Let $\{x_t, t \in I\!R^+\}$ be the continuous time SV model defined by (3.1) and (3.2). For any real h, $h > 0$, define the returns $\{\varepsilon_{th}^{(h)}, t \in \mathbf{N}\}$ by (3.3), let $H_{th}^{(h)}$ be the Hilbert-space generated by $\{1, \varepsilon_{\tau h}^{(h)}, (\varepsilon_{\tau h}^{(h)})^2, \tau \leq t, \tau \in \mathbf{N}\}$ and denote by $h_{(t-1)h}^{(h)}$ the best linear predictor of $(\varepsilon_{th}^{(h)})^2$ given $H_{(t-1)h}^{(h)}$. Then $\varepsilon_{th}^{(h)}$ is a weak GARCH(2,2) and*

$$h_{th}^{(h)} = \omega^{(h)} + \alpha_1^{(h)}(\varepsilon_{th}^{(h)})^2 + \alpha_2^{(h)}(\varepsilon_{(t-1)h}^{(h)})^2 + \beta_1^{(h)} h_{(t-1)h}^{(h)} + \beta_2^{(h)} h_{(t-2)h}^{(h)}, \tag{3.15}$$

with $\omega^{(h)}$ the real number defined in (3.13) and

$$\alpha_1^{(h)} = \gamma_1^{(h)} + \gamma_2^{(h)} - \beta_1^{(h)}, \quad \alpha_2^{(h)} = -\gamma_1^{(h)}\gamma_2^{(h)} - \beta_2^{(h)}$$

$$\beta_1^{(h)} = \frac{\beta_2^{(h)}}{1-\beta_2^{(h)}} \frac{\rho_1^{(h)}}{\rho_2^{(h)}}, \quad \beta_2^{(h)} = \frac{2u^{(h)} + 1 - \sqrt{4u^{(h)}+1}}{2u^{(h)}},$$

where $\gamma_i^{(h)}$ and $\rho_i^{(h)}$, for i=1,2, are defined in (3.8) and (3.14), while $u^{(h)}$ is given by

$$u^{(h)} = 2^{-1}\rho_2(h)^2\rho_1(h)^{-2}\left[-2 - \rho_2(h)^{-1} + sign(\rho_2(h))\sqrt{(2+\rho_2(h)^{-1})^2 - 4\rho_1(h)^2\rho_2(h)^{-2}}\right].$$

3.2 An empirical illustration

Consider a two-factor affine continuous time stochastic volatility model, i.e.,

$$dx_t = \sigma_t dW_t,\ \sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2, d\sigma_{i,t}^2 = k_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_i\sigma_{i,t}dW_{i,t},\ \text{for } i = 1, 2, \tag{3.16}$$

where W_t, $W_{1,t}$, and $W_{2,t}$ are independent standard Brownian processes. The numerical results we will provide are based on the parameter estimates reported in Bollerslev and Zhou (2002) obtained by matching the sample moments of the daily realized volatilities constructed from high-frequency five-minute DM/$ returns spanning from 1986 through 1996 to the corresponding population moments for the integrated volatility. The resulting values are, $k_1 = 0.5708$, $\theta_1 = 0.3257$, $\sigma_1 = 0.2286$, $k_2 = 0.0757$, $\theta_2 = 0.1786$, $\sigma_2 = 0.1096$, implying the existence of a very volatile first factor, along with a much more slowly mean-reverting second factor.

We start our analysis by studying low frequencies, i.e., daily or lower frequencies. More precisely, we provide in Table 1 the weak GARCH(2,2) parameters of returns computed at the following lengths: one day, one week, two weeks, one month, two months, three months, and six months. The results of Table 1 can be summarized as follows. While the model is a (weak) GARCH(2,2) for all frequencies, it appears as a (weak) GARCH(1,1) for the weekly frequency and lower ones. The main reason is that the very volatile but non persistent factor has no impact on volatility clustering. This explains the empirical relevance of the GARCH(1,1) with respect to other ARCH-type models. The second interesting and well-known result is that for very low frequencies, like tri-monthly and lower ones, the ARCH effect is negligible; this result is in line with the ones of, e.g., Diebold (1988), Drost and Nijman (1993), and Meddahi and Renault (2002). Finally, it is clear from Table 1 that some coefficients of the GARCH(2,2) model are negative. We know however since Nelson and Cao (1992) that the positiveness of all coefficients of the GARCH(p,q) model, $p > 1$ or $q > 1$, is not a necessary condition to ensure the positivity of the (weak) GARCH volatility process. In addition, as highlighted by Meddahi and Renault (2002), the weak GARCH volatility process is not necessary a positive process.

Consider now the case of intra-daily frequencies. We report in Table 2 the results for intra-daily returns computed at the following lengths: one day, eight hours, four hours, three hours, one hour, thirty minutes, fifteen minutes, ten minutes, five minutes, one minute, and thirty seconds. We still have some negative coefficients. In addition, when the frequency of observations increases, the persistence of the volatility, measured by the two autoregressive coefficients $\gamma_1(h)$ and $\gamma_2(h)$, increases. Indeed, as pointed out by Nelson (1990), these two autoregressive coefficients become very close to unity.

However, a new result from Table 2 suggests that the moving average roots, i.e. $\lambda_1(h)$ and $\lambda_2(h)$, are also very close to unity and indeed very close to the autoregressive coefficients. This implies that the identification of these parameters from high-frequency returns will be difficult given that the squared returns process is an ARMA process with autoregressive coefficients that are very close to the moving average ones. The identification of the structural parameters, i.e. the parameters of (3.16), will also be difficult for the same reason. Another implication of the closeness of the autoregressive roots and the moving-average ones is that the ARCH effect may disappear. However, it turns out that this not the case. More precisely, Meddahi (2002a) shows that

$$Corr[(\varepsilon_{th}^{(h)})^2, (\varepsilon_{(t-j)h}^{(h)})^2] \to \frac{a_1^2 + a_2^2}{2a_0^2 + 3a_1^2 + 3a_2^2} \text{ when } h \to 0,$$

where $a_0 = \theta_1 + \theta_2$, $a_1 = \; - \sigma_1\sqrt{\theta_1}/\sqrt{2k_1}$ and $a_2 = \; - \sigma_2\sqrt{\theta_2}/\sqrt{2k_2}$. By using the estimates of Bollerslev and Zhou (2002), one gets

$$\frac{a_1^2 + a_2^2}{2a_0^2 + 3a_1^2 + 3a_2^2} = 0.048801,$$

which is small but non-zero. Thus, the ARCH effect on the squared residual process is small, while volatility is very persistent.[9]

[9] It is worth noting that in this analysis, we do not take into account microstructure effects, which is a critical feature of high frequency returns. In particular, as shown by Andersen and Bollerslev (1997), the ARCH effect of returns is substantial when one considers very high frequency returns and that this ARCH effect is mostly due to microstructure effects, like intra-daily periodicities.

4 Conclusion

In this paper, we derived analytically the moving-average roots of a two-factor model. As a result, we characterize the ARMA representation of two-factor models, which are common in financial modeling. This ARMA representation, which is the analytical steady-state of the unobservable variable, like volatility, is useful for filtering, forecasting and statistical inference purposes. As an application, we characterize the weak GARCH(2,2) representation of two-factor continuous time stochastic volatility models.

In Meddahi (2002b), we apply these results to derive the ARMA representation of integrated variance and realized variance. This representation, used in Andersen, Bollerslev and Meddahi (2002), is useful for studying forecasts of integrated and realized variances. Future research will exploit the same representation of realized variance to estimate continuous time stochastic volatility models as in Barndorff-Nielsen and Shephard (2002) and Bollerslev and Zhou (2002).

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Appendix

Proof of (2.5). Let ψ_t be the vector defined by $\psi_t = (v_{1,t}, v_{2,t}, u_t)^\top$. Observe that

$$z_t = A_0\psi_t + A_1\psi_{t-1} + A_2\psi_{t-2}, \text{ where} \tag{A.1}$$

$$A_0 = (0,0,1),\ A_1 = (1,1,-\gamma_1-\gamma_2) \text{ and } A_2 = (-\gamma_1, -\gamma_2, \gamma_1\gamma_2). \tag{A.2}$$

Given that ψ_t is a white noise, we have $Cov[\psi_t, \psi_{t-h}] = 0$ for any $h \neq 0$. Hence, $\forall h,\ \mid h \mid > 2$: $Cov[z_t, z_{t-h}] = Cov[A_0\psi_t + A_1\psi_{t-1} + A_2\psi_{t-2}, A_0\psi_{t-h} + A_1\psi_{t-h-1} + A_2\psi_{t-h-2}] = 0.\blacksquare$

Proof of Proposition 2.1. 1) Given that η_t is a white noise, we have

$$\begin{aligned} Var[z_t] &= (1+\beta_1^2+\beta_2^2)Var[\eta_t], \\ Cov(z_t, z_{t-1}) &= -\beta_1(1-\beta_2)Var[\eta_t], \\ Cov(z_t, z_{t-2}) &= -\beta_2 Var[\eta_t]. \end{aligned} \tag{A.3}$$

Hence, $Var[\eta_t]$ is deduced from the first equality in (A.3). Moreover, since $\rho_2 \neq 0$, the third equality in (A.3) implies that $\beta_2 \neq 0$. Therefore, the second and third equalities in (A.3) imply that $\rho_1{\rho_2}^{-1} = \beta_1(1-\beta_2)\beta_2^{-1}$. Thus, we get β_1 as in (2.12). By combining the first and third equalities of (A.3), we get:

$$-\frac{\beta_2}{\rho_2} = 1 + \frac{\beta_2^2}{(1-\beta_2)^2}\frac{\rho_1^2}{\rho_2^2} + \beta_2^2. \tag{A.4}$$

Define z by $z \equiv \beta_2(1-\beta_2)^{-1}$. Thus, we have $\beta_2 = z(1+z)^{-1}$. Hence, (A.4) becomes

$$-\frac{z}{(1+z)\rho_2} = 1 + z^2\frac{\rho_1^2}{\rho_2^2} + \frac{z^2}{(1+z)^2}.$$

Hence, we get

$$-z(1+z)\frac{1}{\rho_2} = 1 + 2z(1+z) + z^2(1+z)^2\frac{\rho_1^2}{\rho_2^2}, \text{ i.e.}$$

$$s^2\frac{\rho_1^2}{\rho_2^2} + (2+\frac{1}{\rho_2})s + 1 = 0 \quad \text{where} \quad s \equiv z(1+z). \tag{A.5}$$

We now characterize the solutions of (A.5). Their nature, i.e. real or complex, depend on the sign of Δ_s defined by

$$\Delta_s \equiv (2+{\rho_2}^{-1})^2 - 4\rho_1^2\rho_2^{-2}. \tag{A.6}$$

Observe that: $\mid \rho_2 \mid = \mid \beta_2 \mid (1+\beta_1^2+\beta_2^2)^{-1} \leq \mid \beta_2 \mid (1+\beta_2^2)^{-1} \leq 2^{-1}$. Therefore,

$$\begin{aligned} \Delta_s &= 4 + 4\rho_2^{-1} + \rho_2^{-2} - 4\rho_1^2\rho_2^{-2} \geq 4 + 4\rho_2^{-1} + \rho_2^{-2} - 4\rho_2^{-2} = 4 + 4\rho_2^{-1} - 3\rho_2^{-2} \\ &= -3({\rho_2}^{-1} - 2)({\rho_2}^{-1} + 23^{-1}) \geq 0 \quad \text{given that} \ \mid \rho_2 \mid \leq 2^{-1}. \end{aligned}$$

In conclusion, we have $\Delta_s \geq 0$. Therefore, the solutions of (A.5) denoted by s_1 and s_2 are:

$$\begin{aligned} s_1 &= 2^{-1}\rho_2^2\rho_1^{-2}\left[-2-\rho_2^{-1}-\sqrt{(2+\rho_2{}^{-1})^2-4\rho_1^2\rho_2^{-2}}\right], \\ s_2 &= 2^{-1}\rho_2^2\rho_1^{-2}\left[-2-\rho_2^{-1}+\sqrt{(2+\rho_2{}^{-1})^2-4\rho_1^2\rho_2^{-2}}\right]. \end{aligned} \tag{A.7}$$

We have to decide which solution among s_1 and s_2 we have to consider. It is well known that the MA(2) process, admits four equivalent representations corresponding to the polynomials: $(1-\lambda_1 L)(1-\lambda_2 L)$, $(1-\lambda_1^{-1}L)(1-\lambda_2^{-1}L)$, $(1-\lambda_1^{-1}L)(1-\lambda_2 L)$ and $(1-\lambda_1 L)(1-\lambda_2^{-1}L)$. It is easy to show that the first and second representations lead to the same s while the third and fourth ones lead to the same s. It turns out that these two values of s are the solutions of (A.5). Since we are interested in the representation where $\mid \lambda_1 \mid< 1$ and $\mid \lambda_2 \mid< 1$, this mean that the corresponding β_2 is the smallest one in absolute value. Therefore, this is also the case for the corresponding s since s is an increasing function of β_2 given that $s = \beta_2(1-\beta_2)^{-2}$. Let us denote the desired solution by u. Observe that s_1 and s_2 have the same sign, which is also the sign of $-2-\rho_2^{-1}$, i.e. the opposite sign of ρ_2. Besides, $s_1 < s_2$. Therefore, when $\rho_2 < 0$, $u = u_1$ while $u = u_2$ when $\rho_2 > 0$. This is what is stated in (2.14).
We have $u = \beta_2(1-\beta_2)^{-2}$. Therefore, we get

$$u\beta_2^2 - (2u+1)\beta_2 + u = 0.$$

This equation admits two real solutions. The product of the solutions is one. Therefore, to solution is the smallest one in modulus, i.e. (2.13).

Assume now that $\rho_1 = 0$. Then (A.4) becomes

$$\beta_2^2 + \beta_2\rho_2^{-1} + 1 = 0.$$

Observe that the solution of this equation are real since $\frac{1}{\rho_2^2} - 1 \geq 0$; moreover they have the same sign and their product is one. These two roots are given by

$$x_1 = 2^{-1}(-\rho_2^{-1} + \sqrt{\rho_2^{-2} - 4}) \ \text{ and } \ x_2 = 2^{-1}(-\rho_2^{-1} - \sqrt{\rho_2^{-2} - 4}).$$

Observe that $x_2 - x_1 \geq 0$ and, hence, $x_2 \geq x_1$. Thus, when the roots are positive, i.e. $\rho_2 < 0$, β_2 is the smallest root, i.e. $\beta_2 = x_1$; otherwise, $\beta_2 = x_2$. This gives (2.13) when $\rho_1 = 0$.
2) λ_1 and λ_2 are the roots of the equation $\lambda^2 - \beta_1\lambda - \beta_2 = 0$; by solving it, one gets (2.15).■
Proof of Proposition 2.2. Given that ψ_t is a white noise, (A.1) implies that:

$$\begin{aligned} Var[z_t] &= A_0 Var[\psi_t]A_0^\top + A_1 Var[\psi_t]A_1^\top + A_2 Var[\psi_t]A_2^\top \\ Cov[z_t, z_{t-1}] &= A_1 Var[\psi_t]A_0^\top + A_2 Var[\psi_t]A_1^\top \\ Cov[z_t, z_{t-2}] &= A_2 Var[\psi_t]A_0^\top. \end{aligned}$$

Hence, by using (A.2), one gets easily (2.19), (2.20) and (2.21).■

Proof of Proposition 2.3. By using (2.3), one proves easily the proposition.■

Proof of Proposition 3.1. By using Ito's Lemma, we get:

$$(\varepsilon_{th}^{(h)})^2 = \left[\int_{(t-1)h}^{th} \sigma_u dW_u\right]^2 \quad = \int_{(t-1)h}^{th} \sigma_u^2 du + 2\int_{(t-1)h}^{th} (\int_{(t-1)h}^{u} \sigma_s dW_s)\sigma_u dW_u. \tag{A.8}$$

But, we have:

$$\sigma_{i,u}^2 = \theta_i + \exp[-k_i(u-(t-1)h)](\sigma_{1,(t-1)h}^2 - \theta_i)]$$
$$+\exp[-k_i(u-(t-1)h)]\int_{(t-1)h}^{u} \exp[k_i(s-(t-1)h)]\sigma_i\sigma_{i,s}^{\delta_i} dW_{i,s}.$$

Hence, for $i=1,2$, we have:

$$\begin{aligned}\int_{(t-1)h}^{th} \sigma_{i,u}^2 du &= \int_{(t-1)h}^{th} [\theta_1 + \exp[-k_1(u-(t-1)h)](\sigma_{1,(t-1)h}^2 - \theta_1)]du \\ &+ \int_{(t-1)h}^{th} \exp[-k_1(u-(t-1)h)]\left[\int_{(t-1)h}^{u} \exp[k_1(s-(t-1)h)]\sigma_1\sigma_{1,s}^{\delta_1} dW_{1,s}\right] du \\ &= f_{i,(t-1)h}^{(h)} \\ &+ \int_{(t-1)h}^{th} \exp[-k_1(u-(t-1)h)]\left[\int_{(t-1)h}^{u} \exp[k_1(s-(t-1)h)]\sigma_1\sigma_{1,s}^{\delta_1} dW_{1,s}\right] du\end{aligned} \tag{A.9}$$

where $f_{i,(t-1)h}^{(h)}$ is given in (3.5). By combining (A.8) and (A.9), we get (3.4). Observe that

$$\begin{aligned}f_{i,th}^{(h)} &= \theta_i h + \frac{1-\exp[-k_i h]}{k_i}(\sigma_{i,th}^2 - \theta_i) \\ &= \theta_i h + \exp(-k_i h)[f_{i,(t-1)}^{(h)} - \theta_i h] \\ &+ \frac{1-\exp[-k_i h]}{k_i}\exp[-k_i h]\int_{(t-1)h}^{th} \exp[k_i(u-(t-1)h)]\sigma_i\sigma_{i,u}^{\delta_i} dW_{i,u} \\ &= \omega_i^{(h)} + \gamma_i^{(h)} f_{i,(t-1)h}^{(h)} + v_{i,th}^{(h)}\end{aligned}$$

i.e. (3.7), where $\omega_i^{(h)}$ and $\gamma_i^{(h)}$ are given by (3.8) while $v_{i,th}^{(h)}$ is defined in (3.9). It is clear that the $v_{1,th}^{(h)}$, $v_{2,th}^{(h)}$ and $u_{th}^{(h)}$ are martingale difference sequences (m.d.s.). Hence, the vector $(v_{1,th}^{(h)}, v_{2,th}^{(h)}, u_{th}^{(h)})^\top$ is a white noise. In addition, we have:

$$\begin{aligned}Var[v_{i,th}^{(h)}] &= \frac{(1-\exp[-k_i h])^2}{k_i^2}\exp[-2k_i h]\int_{(t-1)h}^{th} \exp[2k_i(u-(t-1)h)]\sigma_i^2 E[\sigma_{i,u}^{2\delta_i}]du \\ &= \frac{(1-\exp[-k_i h])^2}{k_i^2}\exp[-2k_i h]\frac{(\exp[2k_i h]-1)}{2k_i}\sigma_i^2 E[\sigma_{i,u}^{2\delta_i}] \\ &= \frac{(1-\exp[-k_i h])^2(1-\exp[-2k_i h])}{2k_i^3}\sigma_i^2 E[\sigma_{i,u}^{2\delta_i}],\end{aligned}$$

i.e. (3.10).

We will now derive the variance of $u_{th}^{(h)}$, i.e. show (3.11). We have:

$$
\begin{aligned}
Var[u_{th}^{(h)}] &= Var\left[\int_{(t-1)h}^{th} \exp[-k_1(u-(t-1)h)]\left[\int_{(t-1)h}^{u} \exp[k_1(s-(t-1)h)]\sigma_1\sigma_{1,s}^{\delta_1}dW_{1,s}\right]du\right] \\
&+ Var\left[\int_{(t-1)h}^{th} \exp[-k_2(u-(t-1)h)]\left[\int_{(t-1)h}^{u} \exp[k_2(s-(t-1)h)]\sigma_2\sigma_{2,s}^{\delta_2}dW_{2,s}\right]du\right] \\
&+ 4Var\left[\int_{(t-1)h}^{th}(\int_{(t-1)h}^{u}\sigma_s dW_s)\sigma_u dW_u\right].
\end{aligned}
\tag{A.10}
$$

We start by computing the third term in (A.10). Note that:

$$
Var\left[\int_{(t-1)h}^{th}(\int_{(t-1)h}^{u}\sigma_s dW_s)\sigma_u dW_u\right] = E\left[\left(\int_{(t-1)h}^{th}(\int_{(t-1)h}^{u}\sigma_s dW_s)\sigma_u dW_u\right)^2\right].
$$

Define $\tilde{Z}_h$ by

$$
\tilde{Z}_h \equiv \int_0^h(\int_0^u \sigma_s dW_s)\sigma_u dW_u.
$$

By Ito's Lemma, we have:

$$
\tilde{Z}_h^2 = 2\int_0^h \tilde{Z}_u\, d\tilde{Z}_u + \int_0^h d[\tilde{Z},\tilde{Z}]_u = 2\int_0^h \tilde{Z}_u\, (\int_0^u \sigma_s dW_s)\sigma_u dW_u + \int_0^h(\int_0^u \sigma_s dW_s)^2\sigma_u^2 du.
$$

Therefore,

$$
\begin{aligned}
E[\tilde{Z}_h^2] &= E\left[\int_0^h(\int_0^u \sigma_s dW_s)^2\sigma_u^2 du\right] \\
&= \int_0^h E\left[\left(2\int_0^u\left(\int_0^s \sigma_w dW_w\right)\sigma_s dW_s + \int_0^u \sigma_s^2 ds\right)\sigma_u^2 du\right] \\
&= 2\int_0^h E\left[\int_0^u\left(\int_0^s \sigma_w dW_w\right)\sigma_s\sigma_u^2 dW_s\right] + \int_0^h\int_0^u E[\sigma_s^2\sigma_u^2]dsdu \\
&= 2\int_0^h E\left[\int_0^u\left(\int_0^s \sigma_w dW_w\right)\sigma_s\sigma_u^2 dW_s\right] + \int_0^h\int_0^u E[\sigma_s^2\sigma_u^2]dsdu \\
&= 2\int_0^h E\left[\int_0^u\left(\int_0^s \sigma_w dW_w\right)\sigma_s\left(\sum_{i=1}^{2}\theta_i + \exp[-k_i(u-s)])(\sigma_{i,s}^2-\theta_i)\right)dW_s\right] \\
&+ \int_0^h\int_0^u E[\sigma_s^2\sigma_u^2]dsdu
\end{aligned}
$$

given the independence of dW_u with $dW_{1,u}$ and $dW_{2,u}$. Thus,

$$
E[\tilde{Z}_h^2] = \int_0^h\int_0^u E[\sigma_s^2\sigma_u^2]dsdu.
$$

But, we have

$$
\begin{aligned}
E[\sigma_s^2\sigma_u^2] &= Cov[\sigma_s^2,\sigma_u^2] + (E[\sigma_u^2])^2 = Cov[\sigma_{1,s}^2,\sigma_{1,u}^2] + Cov[\sigma_{2,s}^2,\sigma_{2,u}^2] + (\theta_1+\theta_2)^2 \\
&= \exp[-k_1(u-s)]Var[\sigma_{1,u}^2] + \exp[-k_2(u-s)]Var[\sigma_{2,u}^2] + (\theta_1+\theta_2)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E[\tilde{Z}_h^2] &= \sum_{i=1}^{2} \int_0^h \int_0^u \exp[-k_i(u-s)]dsdu \; Var[\sigma_{i,u}^2] + \int_0^h udu \; (\theta_1+\theta_2)^2 \\
&= \sum_{i=1}^{2} \int_0^h \frac{1-\exp[-k_i u]}{k_i} du \; Var[\sigma_{i,u}^2] + \frac{h^2}{2} \; (\theta_1+\theta_2)^2 \\
&= \sum_{i=1}^{2} \frac{\exp[-k_i h] - 1 + k_i h}{k_i^2} \; Var[\sigma_{i,u}^2] + \frac{h^2}{2} \; (\theta_1+\theta_2)^2 .
\end{aligned}
$$

As a summary,

$$
Var\left[\int_{(t-1)h}^{th} (\int_{(t-1)h}^{u} \sigma_s dW_s)\sigma_u dW_u\right] = \sum_{i=1}^{2} \frac{\exp[-k_i h] - 1 + k_i h}{k_i^2} \; Var[\sigma_{i,u}^2] + \frac{h^2}{2} \; (\theta_1+\theta_2)^2 .
$$

We now compute the first and second terms in (A.10). We have:

$$
\int_{(t-1)h}^{th} \exp[-k_i(u-(t-1)h)] \left[\int_{(t-1)h}^{u} \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}\right] du
$$

$$
\begin{aligned}
&= \int_{(t-1)h}^{th} \left[\int_s^{th} \exp[-k_i(u-(t-1)h)]du\right] \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s} \\
&= \int_{(t-1)h}^{th} \frac{(\exp[-k_i(s-(t-1)h)] - \exp[-k_i h])}{k_i} \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s} \\
&= \int_{(t-1)h}^{th} \frac{(1-\exp[k_i(s-th)])}{k_i} \sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}.
\end{aligned}
$$

Thus,

$$
Var\left[\int_{(t-1)h}^{th} \exp[-k_i(u-(t-1)h)] \left[\int_{(t-1)h}^{u} \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}\right] du\right]
$$

$$
\begin{aligned}
&= \int_{(t-1)h}^{th} \frac{(1-\exp[k_i(s-th)])^2}{k_i^2} \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}]ds = \int_0^h \frac{(1-\exp[k_i(s-h)])^2}{k_i^2} ds \; \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}] \\
&= \frac{4\exp[-k_i h] - \exp[-2k_i h] + 2k_i h - 3}{2k_i^3} \; \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}].
\end{aligned}
\tag{A.11}
$$

In conclusion, we have:

$$
\begin{aligned}
Var[u_{th}^{(h)}] &= \sum_{i=1}^{2} \frac{4\exp[-k_i h] - \exp[-2k_i h] + 2k_i h - 3}{2k_i^3} \; \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}] \\
&+ 4\sum_{i=1}^{2} \frac{\exp[-k_i h] - 1 + k_i h}{k_i^2} \; Var[\sigma_{i,u}^2] + 2h^2 \; (\theta_1+\theta_2)^2,
\end{aligned}
$$

i.e.(3.11). The independence of $W_{1,t}$ and $W_{2,t}$ implies that $Cov(v_{1,th}^{(h)}, v_{2,th}^{(h)}) = 0$. In order to achieve the proof of the proposition, we need to compute $Cov[u_{th}^{(h)}, v_{i,th}^{(h)}]$. We have:

$$
\begin{aligned}
Cov[u_{th}^{(h)}, v_{i,th}^{(h)}] = Cov[&\int_{(t-1)h}^{th} \exp[-k_i(u-(t-1)h)] \left[\int_{(t-1)h}^{u} \exp[k_i(s-(t-1)h)]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}\right] du, \\
&\frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \int_{(t-1)h}^{th} \exp[k_i(u-(t-1)h)]\sigma_i \sigma_{i,u}^{\delta_i} dW_{i,u}] \\
= Cov[&\int_0^h \exp[-k_i u] \left[\int_0^u \exp[k_i s]\sigma_i \sigma_{i,s}^{\delta_i} dW_{i,s}\right] du, \\
&\frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \int_0^h \exp[k_i u]\sigma_i \sigma_{i,u}^{\delta_i} dW_{i,u}] \\
= Cov[&\int_0^h \exp[-k_i u] Y_u du, \frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] Y_h]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^h \exp[-k_i u] E[Y_u Y_h] du \ \frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \\
&= \int_0^h \exp[-k_i u] E[Y_u^2] du \ \frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \\
&= \int_0^h \exp[-k_i u] \left(\int_0^u \exp[2k_i s]\sigma_i^2 E[\sigma_{i,s}^{2\delta_i}] ds\right) du \ \frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \\
&= \int_0^h \exp[-k_i u] \frac{(\exp[2k_i u]-1)}{2k_i} du \ \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}] \ \frac{(1-\exp[-k_i h])}{k_i} \exp[-k_i h] \\
&= \frac{(1-\exp[-k_i h])^3}{2k_i^3} \sigma_i^2 E[\sigma_{i,s}^{2\delta_i}]. \blacksquare
\end{aligned}
$$

Proof of Proposition 3.2. By using Meddahi and Renault (2002), we get that $\varepsilon_{th}^{(h)}$ is a SR-SARV(2). Given that we exclude the leverage effect and we assume that the fourth moment of the returns $\varepsilon_{th}^{(h)}$ are finite, we deduce that $\varepsilon_{th}^{(h)}$ is a weak GARCH(2,2) (Meddahi and Renault, 2002). Finally, (3.15) is an application of (2.10) to the variable $(\varepsilon_{th}^{(h)})^2$.$\blacksquare$

Table 1: Weak GARCH(2,2) representation of daily, weekly and monthly returns

d	Freq	$\gamma_1(h)$	$\gamma_2(h)$	$\lambda_1(h)$	$\lambda_2(h)$	$\alpha_1(h)$	$\alpha_2(h)$	$\beta_1(h)$	$\beta_2(h)$
1	1 day	.565	.927	.665	.793	.0337	-.0242	1.46	-.500
5	1 week	.0576	.685	.173	.549	.0198	-.00294	.723	-.0365
10	2 weeks	.00332	.469	.126	.333	.0133	-.000426	.459	-.00113
20	1 month	1.10e-05	.220	.0837	.129	.00713	-4.64e-05	.213	4.40e-05
40	2 months	1.21e-10	.0484	.0218	.0239	.00269	-2.60e-06	.0457	2.60e-06
60	3 months	1.34e-15	.0107	.00463	.00472	.00131	-2.57e-07	.00935	2.57e-07
120	6 months	1.79e-30	.000113	-.0270	-.0244	.0515	-1.60e-07	-.0514	1.60e-07

Table 2: Weak GARCH(2,2) representation of intra-daily returns

1/h	Freq	$\gamma_1(h)$	$\gamma_2(h)$	$\lambda_1(h)$	$\lambda_2(h)$	$\alpha_1(h)$	$\alpha_2(h)$	$\beta_1(h)$	$\beta_2(h)$
1	1 day	.565	.927	.665	.793	.0337	-.0242	1.46	-.500
3	8 hours	.8267	.9751	.8713	.8952	.03536	-.03202	1.766	-.7741
6	4 hours	.9093	.9875	.9280	.9360	.03276	-.03127	1.864	-.8666
8	3 hours	.9311	.9906	.9427	.9479	.03116	-.03011	1.891	-.8923
24	1 hour	.9765	.9969	.9742	.9754	.02376	-.02351	1.950	-.9499
48	30 mn	.9882	.9984	.9835	.9841	.01900	-.01891	1.968	-.9677
96	15 mn	.9941	.9992	.9891	.9894	.01474	-.01470	1.979	-.9786
144	10 mn	.9960	.9995	.9914	.9916	.01255	-.01253	1.983	-.9830
288	5 mn	.9980	.9997	.9941	.9942	.009392	-.009385	1.988	-.9884
1440	1 mn	.9996	.9999	.9975	.9975	.004541	-.004540	1.995	-.9950
2880	30 sec	.9998	.99997	.9982	.9983	.003272	-.003272	1.997	-.9965