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Long-Run Market Configurations in a Dynamic Quality-Ladder Model with Heterogeneity

Mario Samano
Marc Santugini

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Samano: Department of Applied Economics, HEC Montréal
mario.samano@hec.ca
Santugini: Department of Applied Economics and CIRPÉE, HEC Montréal
marc.santugini@hec.ca

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Abstract:
We study the long-run market configurations in a quality-ladder dynamic model. Specifically, we assume that the return to investment in quality differs across the firms. That is, for a given level of investment, one firm has a higher probability to raise the quality of the good it produces. We show that the model can generate five different types of long-run market configurations (market collapse, market collapse or monopoly, monopoly, duopoly and monopoly, and duopoly). A high degree of heterogeneity in the return to investment can mitigate the effect of highly reversible investments on the probability of market collapse, giving rise to non-negligible probabilities of observing a duopoly or even dominance of the firm with the lowest return to investment.

Keywords: Differentiated-good markets, Quality-ladder model, Heterogeneity, Dynamic investment

JEL Classification: C61, C73, L13

1 Introduction

The degree of differentiation among differentiated goods varies greatly among industries. Specifically, differentiated-good markets display a wide variety of long-run market configurations in terms of quality, and thus market shares. For instance, markets for cars or computer processors trade highly differentiated goods. On the other hand, the market for electricity is composed of goods that are not very differentiated.[1] These differences in the degree of differentiation depend on the firms' individual abilities (e.g., firms' expertise) and their willingness (e.g., amount invested in R&D) to improve on the characteristics of their goods. However, it is not clear whether in a strategic setting a firm with a higher likelihood of success of investment will always attain a higher quality level in the long-run.

Recently, Goettler and Gordon [2011] have estimated a dynamic quality-ladder model for the computer processors industry. They find evidence for heterogeneity in the likelihood of success of investment, which can explain differences in the levels of investment and ultimately differences in the levels of quality between the goods.[2] Motivated by this finding, this paper asks the following question. What is the effect of heterogeneity in firms' ability to invest in quality on long-run market configurations? Specifically, under what circumstances are we more likely to observe a situation in which goods are quite differentiated? To answer these questions, we adapt the quality ladder model described in Ericson and Pakes [1995] and the algorithms to numerically solve for its equilibrium such as the one described in McGuire and Pakes [1994] and in a particular case in Levhari and Mirman [1980] to the case of heterogeneous likelihood of success of investment.

Our analysis shows that the dynamic quality-ladder model can generate in the long-run five different distributions on the space of market configurations (market collapse, market collapse and monopoly, monopoly, duopoly and monopoly, and duopoly). As the investment becomes more reversible

[1]Although it is not possible to distinguish the different types of electricity at the moment of utilization, it is possible to distinguish the different market shares according to the source of production: clean vs. dirty sources.

[2]In their adaptation of the Ericson-Pakes model, the source of this heterogeneity in the model is twofold: specific parameters for each firm and the quality distance between the leader and the follower. Specifically, they find a parameter value of 0.0010 for Intel, a value of 0.0019 for AMD. The estimated parameters are different for each firm, capturing the observed heterogeneity of firm dominance in their data.

(i.e., higher depreciation rates), the long-run configurations containing only duopolies become less common in the parameter space of likelihood of investment. If the investment is highly reversible, monopoly configurations become more common but, against usual intuition, the possibility of a duopoly does not completely go away: there is a positive probability that either of the two firms dominates or they coexist. Interestingly, a high degree of heterogeneity can mitigate the effect of highly reversible investments on the probability of market collapse, giving rise to non-negligible probabilities of observing a duopoly or even dominance of the weakest firm. We also show how usual measures of market concentration such as the Herfindahl index cannot reflect much of the observed variation in market structures in this model unless the investment is highly reversible.

We restrict attention to the quality-ladder model without entry or exit. This is not that of a strong assumption since we allow for quality levels of zero yielding zero demand, meaning that the firm producing such good has in exited the market. That however does not prohibit the same firm to become active again in the market if it achieves to increase quality to a positive level. We also note that in our motivating example in Goettler and Gordon [2011], they do not consider entry and exit since the industry they study does not exhibit such behavior during the time window of their data.[3] In another example of the estimation of a quality ladder model, Gowrisankaran and Town [1997] consider the possibility of entry and exit, however all hospitals belong to one of two firm types, and thus if all firms of one type exit, this is equivalent to having quality zero for that type of firm in our model.[4]

Heterogeneity in the Ericson-Pakes dynamic models has been studied in the context of capacity games. Besanko and Doraszelski [2004] conclude that asymmetries of firm size can be due to the effects of price competition leading to long run distributions that exhibit positive probabilities on the monopoly outcomes.[5] Their analysis keeps parameters symmetric across the two firms. We also find such configurations in homogeneous cases, but those configurations are common throughout all the parameter space we consider. The asymmetries in price competition in their model arise because of small

[3]Goettler and Gordon [2011] pp. 1151.

[4]Gowrisankaran and Town [1997] consider two types of hospitals, for-profits and non-for-profits and the ratio between them is endogenous in the model. The parameter governing the probability of success of investment is restricted to be the same for the two hospital types, and yet, the observed market configurations in the data are not symmetric.

[5]This behavior was not found under quantity competition.

asymmetries in capacity accumulation that occur accidentally which makes one firm slightly dominant over the other, making the other firm to give up if investment is highly reversible. In Borkovsky et al. [2010] and Borkovsky et al. [2012], it is shown that the dynamic quality-ladder model can exhibit multiplicity of equilibria even in the absence of entry or exit if the investment is highly permanent. We take a different approach and allow firms to have different parameters in their investment success function and study the limiting distribution over the quality space given the unique equilibrium policies.[6]

The remainder of this article has the following structure. Section 2 introduces the model. In Section 3 we provide computational details and the parametrization of the model. Section 4 presents the results and Section 5 concludes.

2 Model

We introduce heterogeneity in the Ericson-Pakes dynamic quality latter model. We restrict attention to the case of two firms and abstract from entry or exit.[7]

Consider a differentiated-product market in which two firms compete à la Bertrand as well as invest to improve the quality of their products. For $j = 1, 2$, let $\omega_j \in \{0, 1, 2, ..., M\}$ be firm j's quality of the product out of M possible values. Given qualities $\{\omega_1, \omega_2\}$ and prices $\{p_1, p_2\}$, firm j's demand is

$$D\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right) = m \frac{e^{g(\omega_j)-\lambda p_j}}{1 + e^{g(\omega_j)-\lambda p_j} + e^{g(\omega_{3-j})-\lambda p_{3-j}}} \tag{1}$$

where $m > 0$ is the size of the market and

$$g\left(\omega_j\right) = \begin{cases} -\infty & \omega_j = 0 \\ \omega_j, & 1 \leq \omega_j < \omega^* \\ \omega^* + \ln(2 - \exp(\omega^* - \omega_j)), & \omega^* \leq \omega_j \leq M \end{cases} \tag{2}$$

[6] In Borkovsky et al. [2010] figure 5 they provide evidence on the existence of multiple equilibria for depreciation rates below 0.1. Our analysis uses depreciation rates above that level and we check for potential multiplicity of equilibria solving the game in consecutive finite time horizons versions of the model a la Levhari and Mirman [1980].

[7] As discussed in the introduction, one of our two empirical examples in the literature (Goettler and Gordon [2011]) does not consider entry or exit. Moreover, we allow for quality levels of zero and the demand function in this case becomes null, this is equivalent to exiting the market.

maps firm j's product quality into consumer's valuation, $\omega^* \in (0, M]$. Our specification in (2) is similar to Borkovsky et al. (2012) in that $\omega_j = 0$ drives firm j's demand to zero. Although entry or exit are not explicitly modelled, the state $(\omega_1, \omega_2) = (0, 0)$ essentially leads to a temporary collapse of the market.[8] Firm j's instantaneous profits are

$$\pi\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right) = D\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right)\left(p_j - c\right) \tag{3}$$

where $c > 0$ is the constant marginal cost of production. Because market competition has no effect on the dynamics, the pricing game is static. Let $\Pi\left(\omega_j, \omega_{3-j}\right)$ be firm j's instantaneous profit corresponding to the static Bertrand game.[9]

Investment. Each period, firm j invests an amount $x_j \geq 0$ at unit cost $d > 0$ intended to improve product quality. The process for quality is stochastic and subject to an industry-wide shock. Specifically, firm j's product quality evolves stochastically as

$$\tilde{\omega}_j' | \omega_j = \min\{\max\{\omega_j + \tilde{\tau}_j + \tilde{\eta}, 1\}, M\} \tag{4}$$

where $\tilde{\tau}_j$ is a firm-specific shock and $\tilde{\eta}$ is an industry-wide depreciation shock.[10] Each random variable is binary. The firm-specific shock has support $\{0, 1\}$ and depends on the amount of investment, i.e.,

$$\Pr[\tilde{\tau}_j = 1 | x_j] = \frac{\alpha_j x_j}{1 + \alpha_j x_j} = \phi_j(x_j) \tag{5}$$

is firm j's probability of success conditional on investing $x_j \geq 0$. Here, $\alpha_j > 0$ is specific to firm j, which is our only source of parameter heterogeneity. The industry-wide depreciation shock has support $\{-1, 0\}$ such that

$$\Pr[\tilde{\eta} = -1] = \delta \in [0, 1] \tag{6}$$

[8] We call it a temporary collapse of the market since firms can still invest to go back up. In other words, it is possible that for a particular set of parameters even if $(\omega_1, \omega_2) = (0, 0)$, firms' optimal policy functions are positive at that state and they may go back into the game.

[9] That is, for $j = 1, 2$, $\Pi\left(\omega_j, \omega_{3-j}\right) = D\left(p_j^*, p_{3-j}^*; \omega_j, \omega_{3-j}\right)\left(p_j^* - c\right)$ where the pair $\{p_1^*, p_2^*\}$ is the Bertrand equilibrium defined as $p_j^* = \arg\max_{p_j > 0} D_j\left(p_j, p_{3-j}^*; \omega_j, \omega_{3-j}\right)\left(p_j - c\right)$. For all $\{\omega_1, \omega_2\}$, there exists a unique Bertrand-Nash equilibrium (Caplin and Nalebuff [1991]).

[10] A tilde sign distinguishes a random variable from a realization whereas a prime sign indicates a variable in the subsequent period.

is the probability of quality depreciation.[11]

Value Function. Before proceeding with the definition and characterization of the equilibrium, it is useful to write down the firm's value function taking as given the behavior of the other firm. Specifically, for $j = 1, 2$, given x_{3-j}, firm j's infinite-horizon value function satisfies

$$v_j\left(\omega_j, \omega_{3-j}\right) = \max_{x_n \geq 0} \left\{ \Pi\left(\omega_j, \omega_{3-j}\right) - dx_j + \beta \mathbf{E}[v_j(\tilde{\omega}'_j, \tilde{\omega}'_{3-j}) | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \right\} \tag{7}$$

where the expected continuation value function is written as

$$\begin{aligned}
&\mathbf{E}[v_j(\tilde{\omega}'_j, \tilde{\omega}'_{3-j}) | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \\
&= \phi_j(x_j)\phi_{3-j}(x_{3-j}) \cdot \left(\delta v_j\left(\omega_j, \omega_{3-j}\right) + (1-\delta) v_j\left(\omega_j^+, \omega_{3-j}^+\right)\right) \\
&\quad + \phi_j(x_j)(1 - \phi_{3-j}(x_{3-j})) \cdot \left(\delta v_j\left(\omega_j, \omega_{3-j}^-\right) + (1-\delta) v_j\left(\omega_j^+, \omega_{3-j}\right)\right) \\
&\quad + (1 - \phi_j(x_j))\phi_{3-j}(x_{3-j}) \cdot \left(\delta v_j\left(\omega_j^-, \omega_{2-n}\right) + (1-\delta) v_n\left(\omega_n, \omega_{3-j}^+\right)\right) \\
&\quad + (1 - \phi_j(x_j))(1 - \phi_{3-j}(x_{3-j})) \cdot \left(\delta v_j\left(\omega_j^-, \omega_{3-j}^-\right) + (1-\delta) v_j\left(\omega_j, \omega_{3-j}\right)\right)
\end{aligned} \tag{8}$$

with

$$\omega_j^+ \equiv \min\{\omega_j + 1, M\}, \tag{9}$$

$$\omega_{3-j}^+ \equiv \min\{\omega_{3-j} + 1, M\}, \tag{10}$$

$$\omega_j^- \equiv \max\{\omega_j - 1, 0\}, \tag{11}$$

$$\omega_{3-j}^- \equiv \max\{\omega_{3-j} - 1, 0\}. \tag{12}$$

Given an initial state (ω_j, ω_{3-j}) , expression (8) summarizes all possible changes in the states corresponding to investment levels (x_j, x_{3-j}).

Equilibrium. We restrict attention to Markov-perfect equilibrium (MPE) in pure strategies. The pair $\{X_1(\omega_1, \omega_2), X_2(\omega_2, \omega_1)\}$ is an equilibrium if, for $j = 1, 2$, given $X_{3-j}(\omega_{3-j}, \omega_j)$

$$\begin{aligned}
X_j(\omega_j, \omega_{3-j}) = \arg\max_{x_j \geq 0} &\left\{ \Pi\left(\omega_j, \omega_{3-j}\right) - dx_j \right. \\
&\left. + \beta \mathbf{E}[V_j(\tilde{\omega}'_j, \tilde{\omega}'_{3-j}) | \omega_j, \omega_{3-j}, x_j, X_{3-j}(\omega_{3-j}, \omega_j)] \right\}
\end{aligned} \tag{13}$$

[11] The specific values for α_j we use in our simulations lie well within those in the literature (Goettler and Gordon [2011], Gowrisankaran and Town [1997], Borkovsky et al. [2010]).

where for any $(\omega_j, \omega_{3-j}) \in \{0, 1, ..., M\}^2$, the value function satisfies

$$V_j(\omega_j, \omega_{3-j}) = \Pi(\omega_j, \omega_{3-j}) - dX_j(\omega_j, \omega_{3-j})$$
$$+ \beta \mathbf{E}[V_j(\tilde{\omega}_j', \tilde{\omega}_{3-j}')|\omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j)]$$

where, using (9), (10), (11), and (12),

$$\begin{aligned}
&\mathbf{E}[V_j(\tilde{\omega}_j', \tilde{\omega}_{3-j}')|\omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j)] \\
&= \phi_j(X_j(\omega_j, \omega_{3-j}))\phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left(\delta V_j(\omega_j, \omega_{3-j}) + (1-\delta)V_j\left(\omega_j^+, \omega_{3-j}^+\right)\right) \\
&\quad + \phi_j(X_j(\omega_j, \omega_{3-j}))(1 - \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j))) \cdot \left(\delta V_j\left(\omega_j, \omega_{3-j}^-\right) + (1-\delta)V_j\left(\omega_j^+, \omega_{3-j}\right)\right) \\
&\quad + (1 - \phi_j(X_j(\omega_j, \omega_{3-j})))\phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left(\delta V_j\left(\omega_j^-, \omega_{3-j}\right) + (1-\delta)V_j\left(\omega_n, \omega_{3-j}^+\right)\right) \\
&\quad + (1 - \phi_j(X_j(\omega_j, \omega_{3-j})))(1 - \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j))) \cdot \left(\delta V_j\left(\omega_j^-, \omega_{3-j}^-\right) + (1-\delta)V_j(\omega_j, \omega_{3-j})\right).
\end{aligned} \tag{14}$$

The first-order condition and complementary slackness condition are used to characterize the equilibrium. Specifically, for $j = 1, 2$,

$$X_j(\omega_j, \omega_{3-j}) = \max\left\{\frac{-1 + \sqrt{\frac{1}{d}}\sqrt{\frac{\beta\alpha_j}{1+\alpha_{3-j}X_{3-j}(\omega_{3-j},\omega_j)}}\sqrt{\alpha_{3-j}X_{3-j}(\omega_{3-j}, \omega_j)\Delta_j + \Psi_j}}{\alpha_j}, 0\right\} \tag{15}$$

when $\alpha_{3-j}X_{3-j}(\omega_{3-j}, \omega_j)\Delta_j + \Psi_j \geq 0$ and $X_j(\omega_j, \omega_{3-j}) = 0$ otherwise. Here, using (9), (10), (11), and (12),

$$\begin{aligned}
\Delta_j &\equiv \delta\left[V_j(\omega_j, \omega_{3-j}) - V_j\left(\omega_j^-, \omega_{3-j}\right)\right] \\
&\quad + +(1-\delta)\left[V_j\left(\omega_j^+, \omega_{3-j}^+\right) - V_j\left(\omega_j, \omega_{3-j}^+\right)\right],
\end{aligned} \tag{16}$$

$$\begin{aligned}
\Psi_j &\equiv \delta\left[V_j\left(\omega_j, \omega_{3-j}^-\right) - V_j\left(\omega_j^-, \omega_{3-j}^-\right)\right] \\
&\quad + (1-\delta)\left[V_j\left(\omega_j^+, \omega_{3-j}\right) - V_j(\omega_j, \omega_{3-j})\right].
\end{aligned} \tag{17}$$

3 Computation and Parametrization

We use the Pakes-McGuire algorithm to numerically solve for $\{X_1(\omega_1, \omega_2), X_2(\omega_2, \omega_1)\}$ and $\{V_1(\omega_1, \omega_2), V_2(\omega_2, \omega_1)\}$. Since firms are heterogeneous, i.e., $\alpha_1 \neq \alpha_2$, the algorithm consists of iterating on best response operators (Since firms are heterogeneous) until convergence is reached. Specifically, at the initial iteration $\tau = 0$, we set

$$\{X_1^0(\omega_1, \omega_2), X_2^0(\omega_2, \omega_1)\} = \{0, 0\}, \tag{18}$$

with the corresponding value functions is

$$\{V_1^0(\omega_1,\omega_2), V_2^0(\omega_2,\omega_1)\} = \{\Pi(\omega_1,\omega_2), \Pi(\omega_1,\omega_2)\}. \tag{19}$$

For iteration $\tau = 1, 2, ...,$ given $\{X_1^{\tau-1}(\omega_1,\omega_2), X_2^{\tau-1}(\omega_2,\omega_1)\}$ and $\{V_1^{\tau-1}(\omega_1,\omega_2), V_2^{\tau-1}(\omega_2,\omega_1)\}$,

$$X_1^\tau(\omega_1,\omega_2) = \max\left\{\frac{-1+\sqrt{\frac{1}{d}}\sqrt{\frac{\beta\alpha_1}{1+\alpha_2 X_2^{\tau-1}(\omega_2,\omega_1)}}\sqrt{\alpha_2 X_2^{\tau-1}(\omega_2,\omega_1)\,\Delta_1^{\tau-1}+\Psi_1^{\tau-1}}}{\alpha_1}, 0\right\} \tag{20}$$

when $\alpha_2 X_2^{\tau-1}(\omega_2,\omega_1)\,\Delta_1 + \Psi_1 \geq 0$ and $X_1^\tau(\omega_1,\omega_2) = 0$ otherwise, and

$$X_2^\tau(\omega_2,\omega_1) = \max\left\{\frac{-1+\sqrt{\frac{1}{d}}\sqrt{\frac{\beta\alpha_2}{1+\alpha_1 X_1^{\tau-1}(\omega_1,\omega_2)}}\sqrt{\alpha_1 X_1^{\tau-1}(\omega_2,\omega_1)\,\Delta_2^{\tau-1}+\Psi_2^{\tau-1}}}{\alpha_2}, 0\right\} \tag{21}$$

when $\alpha_1 X_1^{\tau-1}(\omega_2,\omega_1)\,\Delta_2^{\tau-1} + \Psi_2^{\tau-1} \geq 0$ and $X_1^\tau(\omega_1,\omega_2) = 0$ otherwise. Here, using (), (), (9), (10), (11), and (12), for $j = 1, 2$,

$$\begin{aligned}
\Delta_j^{\tau-1} &= \delta\left[V_j^{\tau-1}(\omega_j,\omega_{3-j}) - V_j^{\tau-1}(\omega_j-1,\omega_{3-j})\right] \\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j^+,\omega_{3-j}+1\right) - V_j^{\tau-1}(\omega_j,\omega_{3-j}+1)\right], && (22)\\
\Psi_j^{\tau-1} &= \delta\left[V_j^{\tau-1}(\omega_j,\omega_{3-j}-1) - V_j^{\tau-1}(\omega_j-1,\omega_{3-j}-1)\right] && (23)\\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j^+,\omega_{3-j}\right) - V_j^{\tau-1}(\omega_j,\omega_{3-j})\right]. && (24)
\end{aligned}$$

In addition to (20) and (21), the value functions are defined by

$$V_1^\tau(\omega_1,\omega_2) = \Pi(\omega_1,\omega_2) - dX_1^\tau(\omega_1,\omega_2) + \beta\mathbf{E}[V_1^{\tau-1}(\tilde{\omega}_1',\tilde{\omega}_2')|\omega_1,\omega_2,X_1^\tau(\omega_1,\omega_2),X_2^\tau(\omega_2,\omega_1)], \tag{25}$$

$$V_2^\tau(\omega_2,\omega_1) = \Pi(\omega_2,\omega_1) - dX_2^\tau(\omega_2,\omega_1) + \beta\mathbf{E}[V_2^{\tau-1}(\tilde{\omega}_2',\tilde{\omega}_1')|\omega_2,\omega_1,X_2^\tau(\omega_2,\omega_1),X_1^\tau(\omega_1,\omega_2)]. \tag{26}$$

The algorithm stops when some convergence criterion for the value functions and the policy functions are met.

In the PM algorithm, the computed levels of investment at each iteration do not constitute an equilibrium since the best responses (in terms of investment) at iteration τ are in reaction to the investments computed at iteration $\tau - 1$. However, stationary points of such iterations are MPEs. In addition to the PM algorithm, we also apply the algorithm suggested by Levhari and Mirman [1980] (LM) in a resource extraction dynamic game. The algorithm consists of computing the equilibrium for any finite horizon and increasing the horizon (making use of the computation for shorter horizons) until convergence is met. Unlike the PM algorithm, the levels of investment computed under the LM algorithm at each iteration constitutes a Markov-perfect equilibrium. In our numerical analysis, we compute the equilibrium using both algorithms, which always lead to the same converged policy functions. The algorithm that computes the limit of a finite horizon game has been applied in the context of the Ericson-Pakes framework (Goettler and Gordon [2011], and Rand...). A description of the LM algorithm is relegated to the appendix. We note that the PM algorithm is much faster than the LM algorithm. However, the LM algorithm allows us to make sure that reactions function cross at most once.

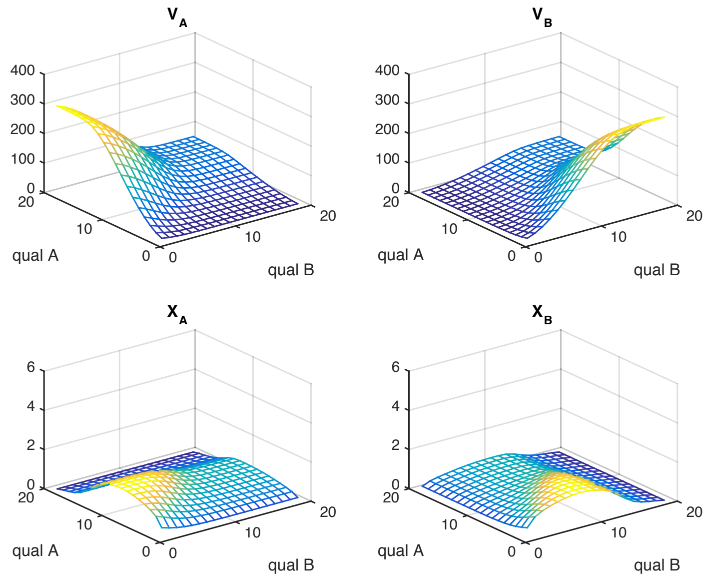
The parameters we use are the same as in Borkovsky et al. [2010] except that we allow for several different values of α.

Table 1: Parameter values

Parameter	M	m	c	ω^*	β	λ	d	α
value	18	5	5	12	0.925	1	1	[0.1, 21]

We first provide evidence on the sensitivity of the model to different values of α for each firm and then proceed with the full analysis of limiting distributions. Figures 1 and 2 show the converged value and policy functions for a homogeneous and a heterogeneous case for some particular parameter values, respectively. When the likelihood of success of investment is the same for both firms, the policy and value functions are identical. However, when this likelihood is not the same across the firms, the firm in disadvantage (the one with a lower α value, firm B in the graph) has to invest more money than the other firm at almost all the states to compensate for this lack of likelihood of success. Because of the low probability of success of increasing its product quality and the higher amount of money spent in the investment, firm B receives in the long run a lower stream of cash flows and ends up having lower values for its value function compared to firm A. This is even true when firm B sells a high quality product and firm A is absent (its quality is equal to 0). The reason for this is that the depreciation effect is strong enough to counteract the possibility of quality improvements, thus leading to low net discounted profits.

Figure 1: Homogeneous case example with $\alpha_A = \alpha_B = 9.7, \delta = 0.5$.



The behavior described above is not an isolated one. In the next section we explore in a systematic manner the consequences of heterogeneity in the likelihood of investment on the long run, that is, if both firms use their equilibrium policy functions to react to each other until there is convergence in the distribution of quality states.

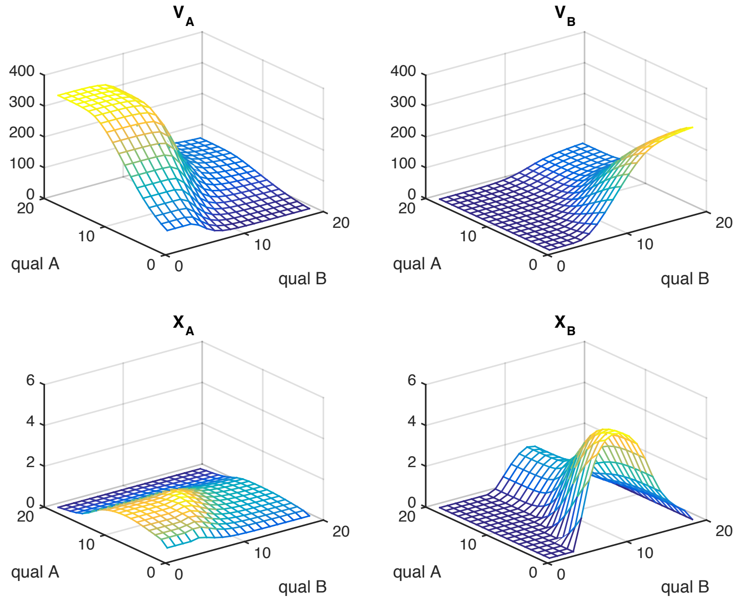
4 Analysis

We provide a numerical analysis of the effect of heterogeneity on the long run market structures.

Let $\pi_t = [\pi_t^0, \ldots, \pi_t^{(M+1)^2}]$ is $(M+1)^2 \times 1$ where π_t^s is the probability that the industry is in state $s = (\omega_j, \omega_k)$ such that $\sum_s \pi_t^s = 1$.

Let $\mathbf{P}$ be a $(M+1)^2 \times (M+1)^2$ transition matrix such that each element provides the probability to transition from one industry state to the other

Figure 2: Heterogeneous case example with $\alpha_A = 18.7, \alpha_B = 0.7, \delta = 0.5$.



one, i.e., $\Pr[(\tilde{\omega}'_H, \tilde{\omega}'_L) \mid (\omega_H, \omega_L)]$. Say something about the sum of the rows.[12]

In general, the transient distribution satisfies

$$\pi_t = \mathbf{P}\pi_{t-1} \tag{27}$$

or

$$\pi_t = \mathbf{P}^t \pi_0 \tag{28}$$

given the initial condition π_0. For each set of paraemters, we use the converged policy functions $x^*_H(\omega)$ to calculate $\mathbf{P}$. In each of the cases we study, there is one eigenvalue equal to zero, the limiting distribution π^* exists and satisfies

$$\pi^* = \mathbf{P}\pi^*. \tag{29}$$

Now, in order to study heterogeneity, we proceed as follows. Let $\alpha_H = \mu + \varepsilon$ and $\alpha_L = \mu - \varepsilon$ such that the distance 2ε reflects the differences in the investment technology to improve quality. Once we obtain the distribution π^*, we count the number of modes. Each of these modes represents the

[12]Appendix provides a detailed derivation of this transition probability.

maximum probability of a specific market configuration.[13] We show that in the long run, the distribution might be unimodal (i.e., only one configuration occurs) or bimodal (two different market configurations are possible) or tri-modal (three different market structures can arise from the same set of parameters).

Possible market configurations. Our model can exhibit five different limiting distributions depending on parameter values. Each one represents a different collection of possible market structures. Those different limiting distributions are: 1) market collapse, 2) market collapse or monopolies, 3) monopolies, 4) duopoly or monopolies, and 5) duopoly. Notice that we do not observe limiting distributions in which only one of the two firms becomes the monopolist with probability 1.

1. The market may collaspe, i.e., quality is driven to zero with probability one and firms do not sell anything.

2. Duopoly with positive quality.

3. One firm may end up dominating, i.e., one firm offers a good of positive quality, i.e., $\omega_j \neq 0$ whereas the other firm offers a good of zero quality, essentially becomes insignificant, i.e., $\omega_k = 0$ For this case, we find that firm H does not always end up dominating the market. Sometimes firm L dominates the market.

4. There is a positive probability for duopoly and for each of the two firms being the monopolist.

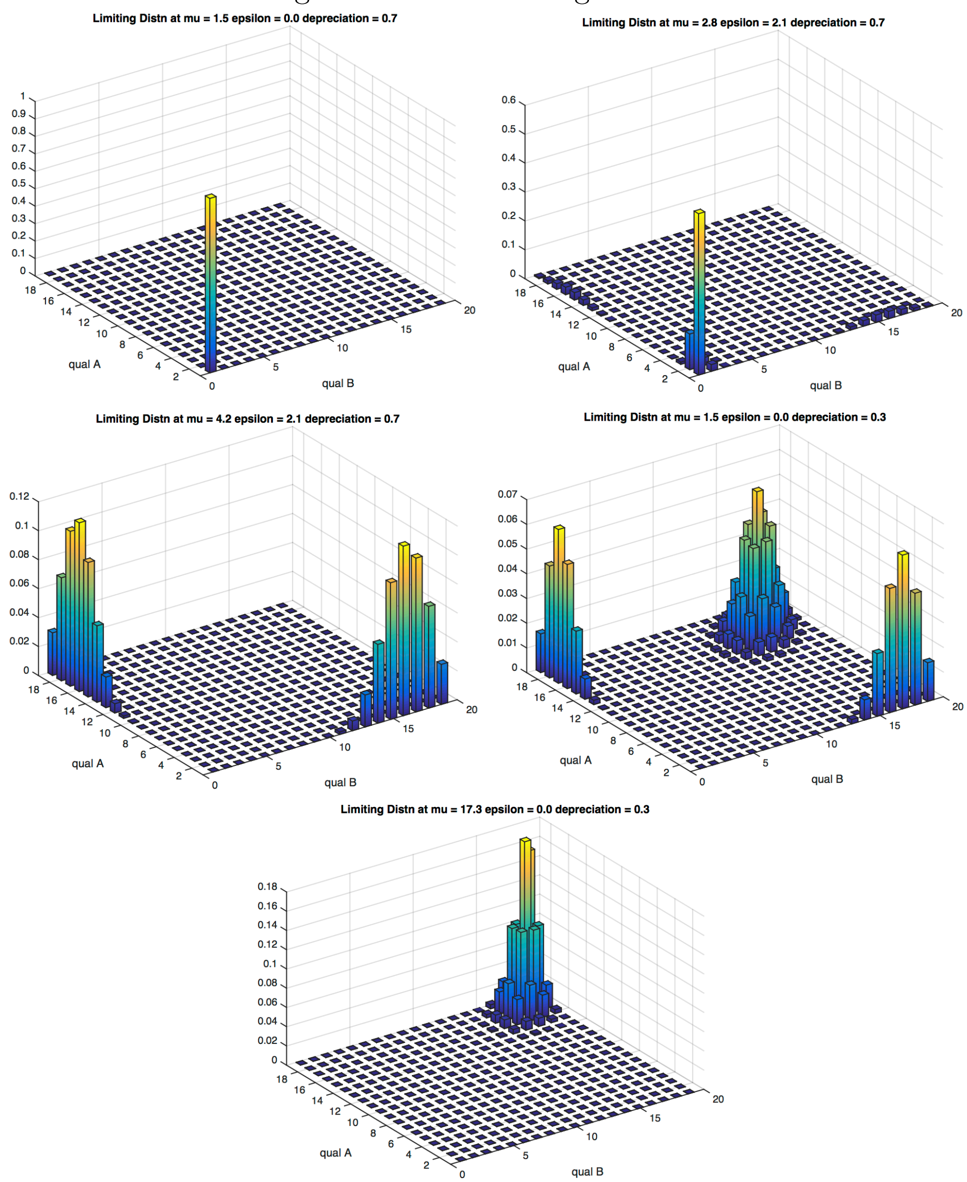
5. All the probability mass is over states where both firms are producing the good with quality greater than zero.

Figure 3 shows an example for each of the cases listed above and their corresponding parameter values.

Market configurations in the parameter space. We investigate the market configurations for different μ, ε, and δ as we increase the rate of depreciation. Figure 6 shows the results. Each panel represents the output for each different depreciation rate we investigated. For each of these panels

[13] We discard modes that have an associated probability of less than 10^{-3}. This threshold is equivalent to discard duopolies that have an associated probability of less than 0.1% chance of occurring.

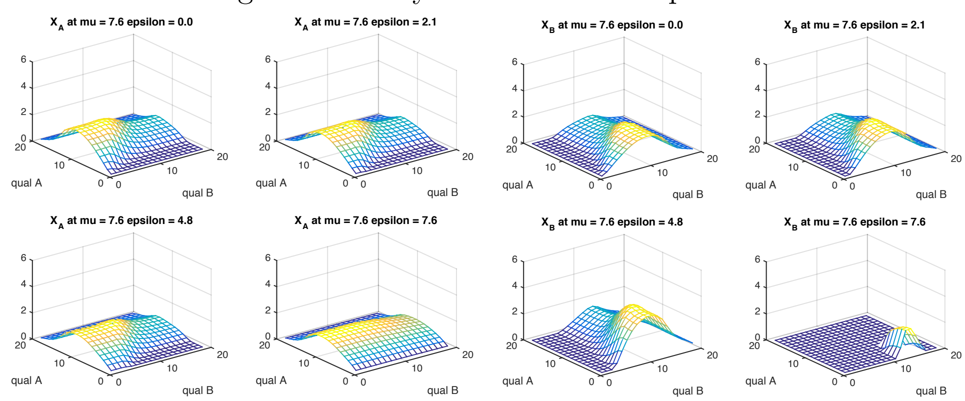
Figure 3: Market configurations.



we show the type of limiting distribution we obtain at each different pair (μ, ϵ). That is, each point represents en entire probability distribution in the long run. Points on the vertical axis represent the cases where both firms are identical. Any point to the right of the vertical axis represents a mean preserving spread of the firm parameters on the likelihood of success of investment, specifically $(\alpha_H, \alpha_L) = (\mu+\epsilon, \mu-\epsilon)$.[14] In other words, the farther to the right from the vertical axis, the higher the degree of heterogeneity.

As investment becomes more reversible (higher depreciation rate) the region for duopoly shrinks from occupying almost the entire set of parameter combinations to no presence at all.

Figure 4: Policy functions low dispersion



The opposite is true for the region that represents a market collapse (both product qualities are zero), it becomes more common as investment becomes highly reversible.

[14] Since below the diagonal the difference $\mu - \epsilon < 0$, none of those points are associated to any model specification and they are left in blank.

Figure 5: Policy functions high dispersion

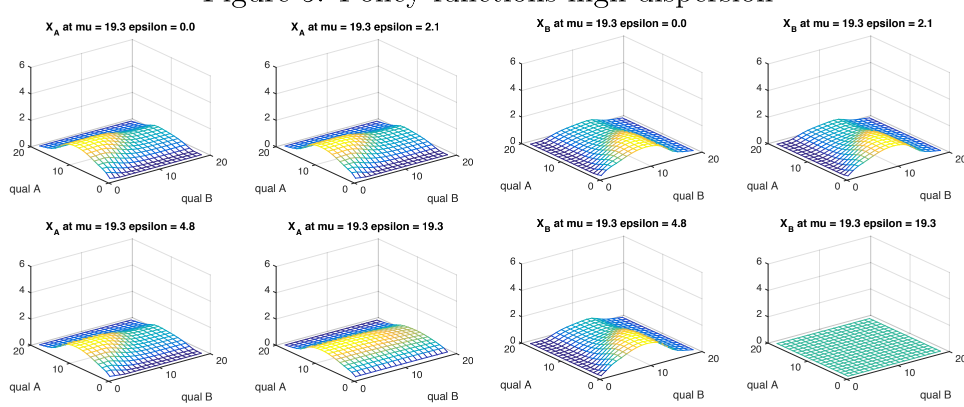
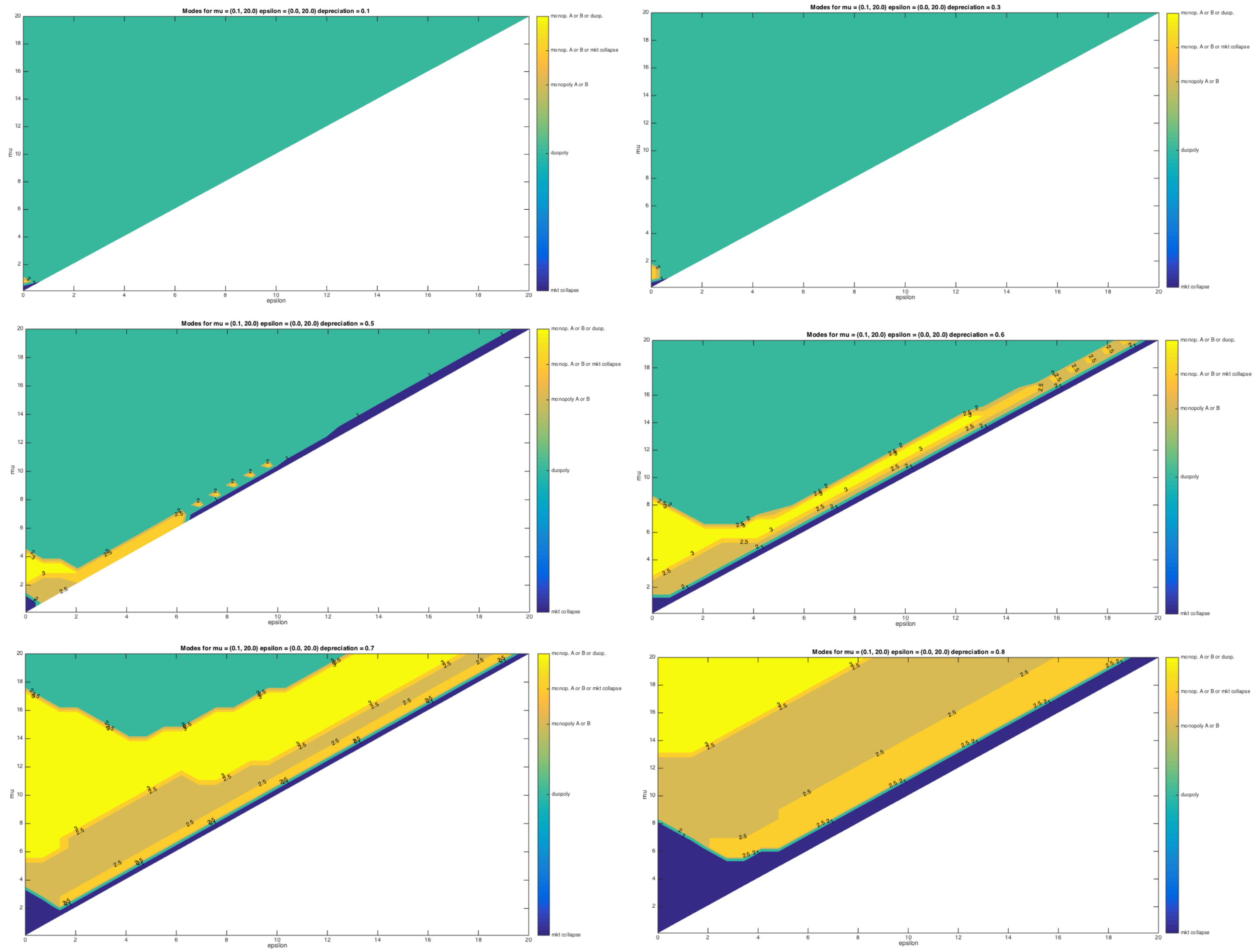


Figure 6: Heterogeneity.



If we fix specific values for mu-epsilon and change the rate of depreciation, we can measure the probability for each market structure. Figure 7 shows the results.

Market concentration. Conditional on having a duopoly structure, we can compute weighted HHIs for each different rate of depreciation and fixed pairs mu-epsilon. For some combinations of parameters, the possibility of converging to a duopoly vanishes and there is no HHI to calculate. For other values, higher rates of depreciation have a negative effect on concentration. Figure 8.

Appendix

LM Algorithm

In this appendix, we explain the Levhari-Mirman (1980) algorithm. We

Value Function, Finite Programs. For $j = 1, 2$, consider firm j's maximization problem for a horizon of τ periods, $\tau = 0, 1, ...$ For $j = 1, 2$, given $x_{3-j} \geq 0$, firm j's value function for a τ-period horizon is

$$v_j^\tau(\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \left\{ \Pi_j(\omega_j, \omega_{3-j}) - d_j x_j + \beta_j \mathbf{E}[v_j^{\tau-1}(\tilde{\omega}_j', \tilde{\omega}_{3-j}') | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \right\} \tag{30}$$

where $\mathbf{E}[\cdot]$ is the expectation operator with respect to $\{\tilde{\omega}_j', \tilde{\omega}_{3-j}'\}$ according to (4), (5), and (6). The value function for the static game (i.e., $\tau = 0$) is

$$v_j^0(\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \{ \Pi_j(\omega_j, \omega_{3-j}) - d_j x_j \}. \tag{31}$$

Consistent with (30), firm j's value function for the infinite-period horizon is thus

$$v_j^\infty(\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \left\{ \Pi_j(\omega_j, \omega_{3-j}) - d_j x_j + \beta_j \mathbf{E}[v_j^\infty(\tilde{\omega}_j', \tilde{\omega}_{3-j}') | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \right\}. \tag{32}$$

Equilibrium. Next, we define the Markov-perfect equilibrium for a game lasting $T + 1$ period, i.e., a horizon of T periods, $T = 0, 1, ..., \infty$. The equilibrium consists of the strategies of the two firms for every horizon from the first period (when there are T periods left) to the last period (when there is no horizon). Condition 1 defines the Nash equilibrium in the static game. Note that in fact, there is no externality since $X_{3-j}^0(\omega_{3-j}, \omega_j)$ has no effect

Figure 7: Probabilities of market structures

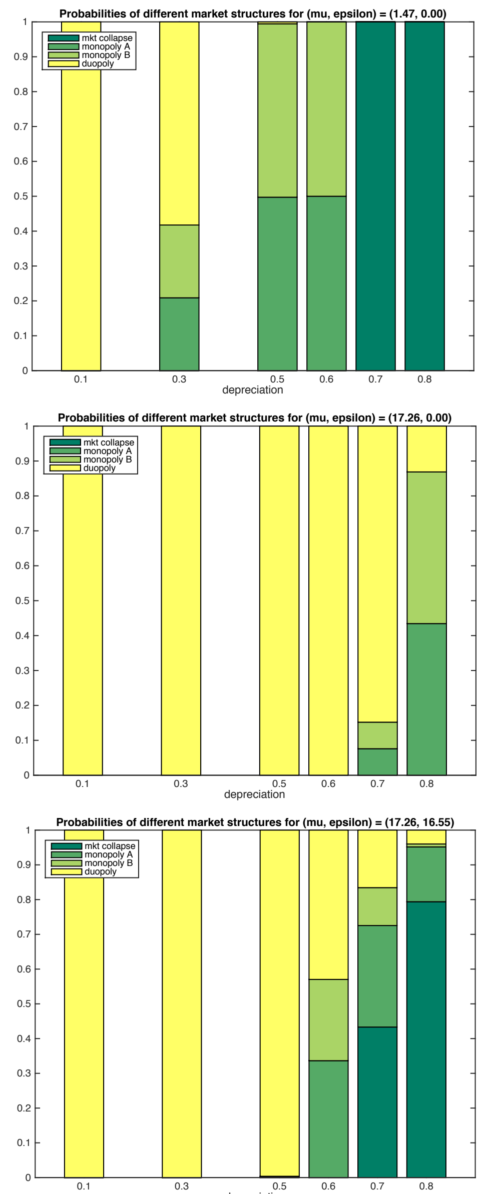
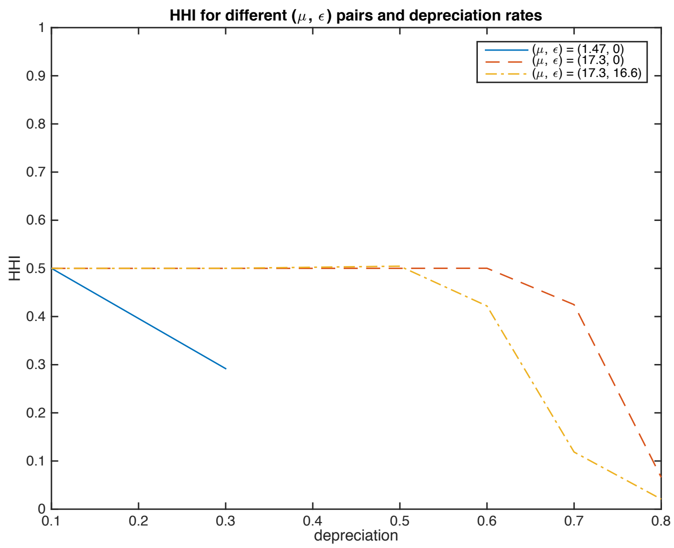


Figure 8: HHI and depreciation



on the zero-period-horizon objective function for firm j. Condition 2 states the equilibrium for every higher horizon of the game. For $\tau = 1, 2, 3, ..., T$, expressions (35) and (36) reflect the recursive nature of the equilibrium in which the equilibrium continuation value function for a $\tau - 1$-period horizon depends on the equilibrium strategies for τ'-period horizons, $\tau - 1 > \tau' \geq 0$.

Definition 1 *The tuple* $\{X_1^\tau(\omega_1, \omega_2), X_2^\tau(\omega_2, \omega_1)\}_{\tau=0}^T$ *is a Markov-perfect Nash equilibrium for a game of T-period horizons if, for all* $\{\omega_1, \omega_2\}$,

1. *For* $\tau = 0$, *for* $j = 1, 2$, *given* $X_{3-j}^0(\omega_{3-j}, \omega_j)$,

$$X_j^0(\omega_{3-j}, \omega_j) = \arg\max_{x_j \geq 0} \left\{ \Pi_j \left(\omega_j, \omega_{3-j} \right) - d_j x_j \right\}. \tag{33}$$

2. *For* $\tau = 1, 2, \ldots, T$, *for* $j = 1, 2$, *given* $X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)$ *and* $\{X^t_1(\omega_1, \omega_2), X^t_2(\omega_2, \omega_1)\}^{\tau-1}_{t=0}$,

$$
\begin{aligned}
&X^{\tau}_j(\omega_{3-j}, \omega_j) \\
&= \arg\max_{x_j \geq 0} \{\Pi_j(\omega_j, \omega_{3-j}) - d_j x_j \\
&+ \beta_j \phi_j(x_j) \phi_{3-j}(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left(\delta V^{\tau-1}_j(\omega_j, \omega_{3-j}) + (1-\delta) V^{\tau-1}_j(\omega_j + 1, \omega_{3-j} + 1)\right) \\
&+ \beta_j \phi_j(x_j)\left(1 - \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right)\right) \cdot \left(\delta V^{\tau-1}_j(\omega_j, \omega_{3-j} - 1) + (1-\delta) V^{\tau-1}_j(\omega_j + 1, \omega_{3-j})\right) \\
&+ \beta_j (1 - \phi_j(x_j)) \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right) \cdot \left(\delta V^{\tau-1}_j(\omega_j - 1, \omega_{3-j}) + (1-\delta) V^{\tau-1}_j(\omega_j, \omega_{3-j} + 1)\right) \\
&+ \beta_j (1 - \phi_j(x_j))(1 - \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right)) \cdot \left(\delta V^{\tau-1}_j(\omega_j - 1, \omega_{3-j} - 1) + (1-\delta) V^{\tau-1}_j(\omega_j, \omega_{3-}\right.
\end{aligned}
\tag{34}
$$

where, for any $y, z \in \{1, 2, ..., M\}$,

$$
V^{\tau'-1}_j(y, z) = \begin{cases} \Pi_j(y, z) - d_j X^0_j(y, z) & \tau' = 1 \\ \Pi_j(y, z) - d_j X^{\tau'-1}_j(y, z) + \beta_j \cdot \Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y)) & \tau' = 2, 3, ..., T \end{cases}
\tag{35}
$$

is the value function for a $\tau' - 1$ *period horizon for any state vector* (y, z) *with*

$$
\begin{aligned}
&\Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y)) \\
&= \phi_j(X^{\tau'-1}_j(y, z)) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y) \cdot \left(\delta V^{\tau'-2}_j(y, z) + (1-\delta) V^{\tau'-2}_j(y + 1, z + 1)\right) \\
&+ \phi_j(X^{\tau'-1}_j(y, z))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y))) \cdot \left(\delta V^{\tau'-2}_j(y, z - 1) + (1-\delta) V^{\tau'-2}_j(y + 1, z)\right) \\
&+ (1 - \phi_j(X^{\tau'-1}_j(y, z))) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left(\delta V^{\tau'-2}_j(y - 1, z) + (1-\delta) V^{\tau'-2}_j(y, z + 1)\right) \\
&+ (1 - \phi_j(X^{\tau'-1}_j(y, z)))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left(\delta V^{\tau'-2}_j(y - 1, z - 1) + (1-\delta) V^{\tau'-2}_j(y, z)\right)
\end{aligned}
\tag{36}
$$

is the expected continuation value function corresponding to the equilibrium for a horizon of $\tau' - 2$ *periods.*

Proposition states the Markov-perfect Nash equilibrium for each horizon of the game.

Proposition 2 *Consider a game of* T*-period horizons.*

1. *For* $\tau = 0$,

$$
\left\{X^0_1(\omega_1, \omega_2), X^0_2(\omega_2, \omega_1)\right\} = \{0, 0\},
\tag{37}
$$

with the corresponding value function is

$$V_j^0\left(\omega_j, \omega_{3-j}\right) = \Pi_j\left(\omega_j, \omega_{3-j}\right). \tag{38}$$

2. *For* $\tau \geq 1$*, given* $\{V_1^{\tau-1}\left(\omega_1, \omega_2\right), V_2^{\tau-1}(\omega_2, \omega_1)$*,* $\{X_1^{\tau}\left(\omega_1, \omega_2\right), X_2^{\tau}\left(\omega_1, \omega_2\right)\}$ *is defined by*

$$X_1^{\tau}\left(\omega_1, \omega_2\right) = \max\left\{\frac{-1 + \sqrt{\frac{1}{d_1}}\sqrt{\frac{\beta_1\alpha_1}{1+\alpha_2 X_2^{\tau}(\omega_2,\omega_1)}}\sqrt{\alpha_2 X_2^{\tau}\left(\omega_2, \omega_1\right)\Delta_1^{\tau-1} + \Psi_1^{\tau-1}}}{\alpha_1}, 0\right\}, \tag{39}$$

$$X_2^{\tau}\left(\omega_2, \omega_1\right) = \max\left\{\frac{-1 + \sqrt{\frac{1}{d_2}}\sqrt{\frac{\beta_2\alpha_2}{1+\alpha_1 X_1^{\tau}(\omega_1,\omega_2)}}\sqrt{\alpha_1 X_1^{\tau}\left(\omega_1, \omega_2\right)\Delta_2^{\tau-1} + \Psi_2^{\tau-1}}}{\alpha_2}, 0\right\}, \tag{40}$$

where for $j = 1, 2$*,*

$$\begin{aligned}
\Delta_j^{\tau-1} &\equiv \delta\left[V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right) - V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j}\right)\right] \\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j} + 1\right) - V_j^{\tau-1}\left(\omega_j, \omega_{3-j} + 1\right)\right], \qquad (41) \\
\Psi_j^{\tau-1} &\equiv \delta\left[V_j^{\tau-1}\left(\omega_j, \omega_{3-j} - 1\right) - V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j} - 1\right)\right] \\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j}\right) - V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right)\right]. \qquad (42)
\end{aligned}$$

Proof. The first-order condition corresponding to (34) is

$$\begin{aligned}
&- d_j + \beta_j \frac{\alpha_j}{\left(1+\alpha_j x_j\right)^2}\phi_{3-j}(X_{3-j}^{\tau}(\omega_{3-j}, \omega_j)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j} + 1\right)\right) \\
&+ \beta_j \frac{\alpha_j}{\left(1+\alpha_j x_j\right)^2}(1 - \phi_{3-j}\left(X_{3-j}^{\tau}(\omega_{3-j}, \omega_j)\right)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j} - 1\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j}\right)\right) \\
&- \beta_j \frac{\alpha_j}{\left(1+\alpha_j x_j\right)^2}\phi_{3-j}\left(X_{3-j}^{\tau}(\omega_{3-j}, \omega_j)\right) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j}\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j, \omega_{3-j} + 1\right)\right) \\
&-\beta_j \frac{\alpha_j}{\left(1+\alpha_j x_j\right)^2}(1 - \phi_{3-j}\left(X_{3-j}^{\tau}(\omega_{3-j}, \omega_j)\right)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j} - 1\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right)\right) \qquad (43) \\
&= 0 \qquad (44)
\end{aligned}$$

which yields (15) and thus (40), as long as the second-order condition is satisfied, i.e., for $j, 3-j = 1, 2, j \neq 3-j$,

$$
\begin{aligned}
&-\beta_j \frac{2\alpha_j^2}{\left(1+\alpha_j x_j\right)^3} \frac{\alpha_{3-j} x_{3-j}}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right) + (1-\delta) V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j} + 1\right)\right) \\
&-\beta_j \frac{2\alpha_j^2}{\left(1+\alpha_j x_j\right)^3} \frac{1}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j} - 1\right) + (1-\delta) V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j}\right)\right) \\
&+\beta_j \frac{2\alpha_j^2}{\left(1+\alpha_j x_j\right)^3} \frac{\alpha_{3-j} x_{3-j}}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j}\right) + (1-\delta) V_j^{\tau-1}\left(\omega_j, \omega_{3-j} + 1\right)\right) \\
&+\beta_j \frac{2\alpha_j^2}{\left(1+\alpha_j x_j\right)^3} \frac{1}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j} - 1\right) + (1-\delta) V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right)\right) < 0\,.
\end{aligned}
\tag{45}
$$

■

Algorithm. Having described the model and define the equilibrium. We now proceed with the characterization of the MPE. Here, we solve the equilibrium recursively as in Levhari and Mirman (1980). Consider first the static game of investment, i.e., $\tau = 0$. Then, there is no externality, and no firm has an incentive to invest, i.e., the Markov-perfect equilibrium for a game of 0-period horizon is simply

$$
\left\{X_1^1\left(\omega_1, \omega_2\right), X_2^1\left(\omega_1, \omega_2\right)\right\} = \{0, 0\}\,, \tag{46}
$$

with the corresponding value function is

$$
V_j^0\left(\omega_j, \omega_{3-j}\right) = \Pi_j\left(\omega_j, \omega_{3-j}\right). \tag{47}
$$

Hence, there is a unique equilibrium for the no-horizon game in which the firms do not invest and the value function is equal to the profit function corresponding to the Bertrand game.

Consistent with the solution of the equilibrium, we characterize the equilibrium for each horizon. Each iteration is an horizon with the caveat that at each iteration, the solution to the reaction function is a Markov-perfect Nash equilibrium (and not an approximation). Hence, wherever we stop, we have an equilibrium. The question remains whether we converge to the stationary Markov-perfect Nash equilibrium (in infinite horizons).

1. For $\tau = 0$,

$$
\left\{X_1^0\left(\omega_1, \omega_2\right), X_2^0\left(\omega_2, \omega_1\right)\right\} = \{0, 0\}\,, \tag{48}
$$

with the corresponding value function is

$$V_j^0(\omega_j, \omega_{3-j}) = \Pi_j(\omega_j, \omega_{3-j}). \tag{49}$$

2. For $\tau \geq 1$, given $\{V_1^{\tau-1}(\omega_1, \omega_2), V_2^{\tau-1}(\omega_2, \omega_1)\}$, firm j's reaction function

$$R_1^\tau(x_2) = \max\left\{\frac{-1 + \sqrt{\frac{1}{d_1}}\sqrt{\frac{\beta_1\alpha_1}{1+\alpha_2 x_2}}\sqrt{\alpha_2 x_2 \Delta_1^{\tau-1} + \Psi_1^{\tau-1}}}{\alpha_1}, 0\right\}, \tag{50}$$

$$R_2^\tau(x_1) = \max\left\{\frac{-1 + \sqrt{\frac{1}{d_2}}\sqrt{\frac{\beta_2\alpha_2}{1+\alpha_1 x_1}}\sqrt{\alpha_1 x_1 \Delta_2^{\tau-1} + \Psi_2^{\tau-1}}}{\alpha_2}, 0\right\} \tag{51}$$

where for $j, 3-j = 1, 2, j \neq 3-j$,

$$\begin{aligned}
\Delta_j^{\tau-1} \equiv\ & \delta\left[V_j^{\tau-1}(\omega_j, \omega_{3-j}) - V_j^{\tau-1}(\omega_j - 1, \omega_{3-j})\right] \\
& + (1-\delta)\left[V_j^{\tau-1}(\omega_j + 1, \omega_{3-j} + 1) - V_j^{\tau-1}(\omega_j, \omega_{3-j} + 1)\right], \qquad (52) \\
\Psi_j^{\tau-1} \equiv\ & \delta\left[V_j^{\tau-1}(\omega_j, \omega_{3-j} - 1) - V_j^{\tau-1}(\omega_j - 1, \omega_{3-j} - 1)\right] \\
& + (1-\delta)\left[V_j^{\tau-1}(\omega_j + 1, \omega_{3-j}) - V_j^{\tau-1}(\omega_j, \omega_{3-j})\right] \qquad (53)
\end{aligned}$$

where

$$V_j^{\tau-1}(y,z) = \begin{cases} \Pi_j(y,z) - d_j X_j^0(y,z) & \tau = 1 \\ \Pi_j(y,z) - d_j X_j^{\tau-1}(y,z) + \beta_j \cdot \Gamma_j^{\tau-2}(X_j^{\tau'-1}(y,z), X_{3-j}^{\tau'-1}(z,y)) & \tau = 2, 3, ..., T \end{cases} \tag{54}$$

is the value function for a $\tau - 1$ period horizon for any state vector (y, z) with

$$\begin{aligned}
&\Gamma_j^{\tau-2}(X_j^{\tau-1}(y,z), X_{3-j}^{\tau-1}(z,y)) \\
&= \phi_j(X_j^{\tau'-1}(y,z))\phi_{3-j}(X_{3-j}^{\tau'-1}(z,y)) \cdot \left(\delta V_j^{\tau'-2}(y,z) + (1-\delta)V_j^{\tau'-2}(y+1, z+1)\right) \\
&+ \phi_j(X_j^{\tau'-1}(y,z))(1 - \phi_{3-j}(X_{3-j}^{\tau'-1}(z,y))) \cdot \left(\delta V_j^{\tau'-2}(y, z-1) + (1-\delta)V_j^{\tau'-2}(y+1, z)\right) \\
&+ (1 - \phi_j(X_j^{\tau'-1}(y,z)))\phi_{3-j}(X_{3-j}^{\tau'-1}(z,y)) \cdot \left(\delta V_j^{\tau'-2}(y-1, z) + (1-\delta)V_j^{\tau'-2}(y, z+1)\right) \\
&+ (1 - \phi_j(X_j^{\tau'-1}(y,z)))(1 - \phi_{3-j}(X_{3-j}^{\tau'-1}(z,y)) \cdot \left(\delta V_j^{\tau'-2}(y-1, z-1) + (1-\delta)V_j^{\tau'-2}(y,z)\right)
\end{aligned} \tag{55}$$

Transition Probability Matrix

For $j = 1, 2$, let $\omega_j \in \{0, 1, 2, ..., M_j\}$. Using the converged policy functions, for $j = 1, 2$,

$$\tilde{\omega}_j' | \omega_j = \min\{\max\{\omega_j + \tilde{\tau}_j + \tilde{\eta}, 1\}, M\} \tag{56}$$

where $\tau_j \in \{1, 0\}$ such that $\Pr[\tilde{\tau}_j = 1] = \phi_j(\omega_1, \omega_2) = \frac{\alpha_j X_j(\omega_j, \omega_{3-j})}{1+\alpha_j X_j(\omega_j, \omega_{3-j})}$ and $\eta \in \{-1, 0\}$ such that $\Pr[\tilde{\eta} = -1] = \delta$.

We want to calculate all transition probabilities such as $\Pr[(\omega_1', \omega_2') \,|\, (\omega_1, \omega_2)]$. We consider each case separately.

1. Suppose that (ω_1, ω_2) is such that $\omega_1, \omega_2 \notin \{0, M\}$. Given (ω_1, ω_2), there are $(M + 1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}\Pr[(\omega_1, \omega_2) \,|\, (\omega_1, \omega_2)] &= \delta\phi_1(\omega_1, \omega_2)\,\phi_2(\omega_2, \omega_1) \\ &\quad + (1-\delta)(1-\phi_1(\omega_1, \omega_2))(1-\phi_2(\omega_2, \omega_1)),\end{aligned} \tag{57}$$

$$\Pr[(\omega_1 + 1, \omega_2) \,|\, (\omega_1, \omega_2)] = (1-\delta)\phi_1(\omega_1, \omega_2)(1-\phi_2(\omega_2, \omega_1)), \tag{58}$$

$$\Pr[(\omega_1 - 1, \omega_2) \,|\, (\omega_1, \omega_2)] = \delta(1-\phi_1(\omega_1, \omega_2))\,\phi_2(\omega_2, \omega_1), \tag{59}$$

$$\Pr[(\omega_1, \omega_2 - 1) \,|\, (\omega_1, \omega_2)] = \delta\phi_1(\omega_1, \omega_2)(1-\phi_2(\omega_2, \omega_1)), \tag{60}$$

$$\Pr[(\omega_1 - 1, \omega_2 - 1) \,|\, (\omega_1, \omega_2)] = \delta(1-\phi_1(\omega_1, \omega_2))(1-\phi_2(\omega_2, \omega_1)), \tag{61}$$

$$\Pr[(\omega_1, \omega_2 + 1) \,|\, (\omega_1, \omega_2)] = (1-\delta)(1-\phi_1(\omega_1, \omega_2))\,\phi_2(\omega_2, \omega_1), \tag{62}$$

$$\Pr[(\omega_1 + 1, \omega_2 + 1) \,|\, (\omega_1, \omega_2)] = (1-\delta)\,\phi_1(\omega_1, \omega_2)\phi_2(\omega_2, \omega_1). \tag{63}$$

Given these above probabilities and the ones equal to zero, they sum up to one, I checked.

2. Suppose that $(\omega_1, \omega_2) = (0, 0)$. Given (ω_1, ω_2), there are $(M + 1)^2$ conditional probabilities to calculate. All of them are zero except

$$\Pr[(0,0)\,|(0,0)] = 1 - (1-\delta)(\phi_1(0,0) + \phi_2(0,0) - \phi_1(0,0)\phi_2(0,0)), \tag{64}$$

$$\Pr[(1,0)\,|(0,0)] = (1-\delta)\phi_1(0,0)(1-\phi_2(0,0)), \tag{65}$$

$$\Pr[(0,1)\,|(0,0)] = (1-\delta)(1-\phi_1(0,0))\,\phi_2(0,0), \tag{66}$$

$$\Pr[(1,1)\,|(0,0)] = (1-\delta)\,\phi_1(0,0)\phi_2(0,0). \tag{67}$$

3. Suppose that $(\omega_1, \omega_2) = (M, M)$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\Pr[(M, M) \,|\, (M, M)] = 1 - \delta(1 - \phi_1(M, M)\,\phi_2(M, M)), \tag{68}$$

$$\Pr[(M-1, M) \,|\, (M, M)] = \delta\,(1 - \phi_1(M, M))\,\phi_2(M, M)\,, \tag{69}$$

$$\Pr[(M, M-1) \,|\, (M, M)] = \delta\phi_1(M, M)\,(1 - \phi_2(M, M))\,, \tag{70}$$

$$\Pr[(M-1, M-1) \,|\, (M, M)] = \delta\,(1 - \phi_1(M, M))\,(1 - \phi_2(M, M))\,. \tag{71}$$

4. Suppose that $(\omega_1, \omega_2) = (0, M)$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\Pr[(0, M) \,|\, (0, M)] = 1 - (1-\delta)\phi_1(0, M) - \delta\,(1 - \phi_2(0, M))\,, \tag{72}$$

$$\Pr[(1, M) \,|\, (0, M)] = (1-\delta)\phi_1(0, M)\,, \tag{73}$$

$$\Pr[(0, M-1) \,|\, (0, M)] = \delta\,(1 - \phi_2(0, M))\,. \tag{74}$$

5. Suppose that $(\omega_1, \omega_2) = (M, 0)$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\Pr[(M, 0) \,|\, (M, 0)] = 1 - (1-\delta)\phi_2(0, M) - \delta\,(1 - \phi_1(M, 0))\,, \tag{75}$$

$$\Pr[(M, 1) \,|\, (M, 0)] = (1-\delta)\phi_2(0, M)\,, \tag{76}$$

$$\Pr[(M-1, 0) \,|\, (M, 0)] = \delta\,(1 - \phi_1(M, 0))\,. \tag{77}$$

6. Suppose that (ω_1, ω_2) is such that $\omega_1 = 0$ and $\omega_2 \notin \{0, M\}$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}\Pr[(0, \omega_2) \,|\, (0, \omega_2)] = {} & \delta\phi_2(\omega_2, 0) \\ & + (1-\delta)\,(1 - \phi_1(0, \omega_2))\,(1 - \phi_2(\omega_2, 0))\,, \end{aligned} \tag{78}$$

$$\Pr[(1, \omega_2) \,|\, (0, \omega_2)] = (1-\delta)\phi_1(0, \omega_2)\,(1 - \phi_2(\omega_2, 0))\,, \tag{79}$$

$$\Pr[(0, \omega_2 - 1) \,|\, (0, \omega_2)] = \delta\,(1 - \phi_2(\omega_2, 0))\,, \tag{80}$$

$$\Pr[(0, \omega_2 + 1) \,|\, (0, \omega_2)] = (1-\delta)\,(1 - \phi_1(0, \omega_2))\,\phi_2(\omega_2, 0)\,, \tag{81}$$

$$\Pr[(1, \omega_2 + 1) \,|\, (0, \omega_2)] = (1-\delta)\,\phi_1(0, \omega_2)\,\phi_2(\omega_2, 0)\,. \tag{82}$$

7. Suppose that (ω_1, ω_2) is such that $\omega_1 \notin \{0, M\}$ and $\omega_2 = 0$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(\omega_1, 0) \,|\, (\omega_1, 0)] &= \delta\phi_1(\omega_1, 0) \\
&\quad + (1-\delta)(1-\phi_2(0, \omega_1))(1-\phi_1(\omega_1, 0)), \quad (83) \\
\Pr[(\omega_1, 1) \,|\, (\omega_1, 0)] &= (1-\delta)\phi_2(0, \omega_1)(1-\phi_1(\omega_1, 0)), \quad (84) \\
\Pr[(\omega_1 - 1, 0) \,|\, (\omega_1, 0)] &= \delta(1-\phi_1(\omega_1, 0)), \quad (85) \\
\Pr[(\omega_1 + 1, 0) \,|\, (\omega_1, 0)] &= (1-\delta)(1-\phi_2(0, \omega_1))\phi_1(\omega_1, 0), \quad (86) \\
\Pr[(\omega_1 + 1, 1) \,|\, (\omega_1, 0)] &= (1-\delta)\phi_2(0, \omega_1)\phi_1(\omega_1, 0). \quad (87)
\end{aligned}$$

8. Suppose that (ω_1, ω_2) is such that $\omega_1 = M$ and $\omega_2 \notin \{0, M\}$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(M, \omega_2) \,|\, (M, \omega_2)] &= \delta\phi_1(M, \omega_2)\phi_2(\omega_2, M) \\
&\quad + (1-\delta)(1-\phi_2(\omega_2, M)) \quad (88) \\
\Pr[(M-1, \omega_2) \,|\, (M, \omega_2)] &= \delta(1-\phi_1(M, \omega_2))\phi_2(\omega_2, M) \quad (89) \\
\Pr[(M, \omega_2 - 1) \,|\, (M, \omega_2)] &= \delta\phi_1(M, \omega_2)(1-\phi_2(\omega_2, M)) \quad (90) \\
\Pr[(M-1, \omega_2 - 1) \,|\, (M, \omega_2)] &= \delta(1-\phi_1(M, \omega_2))(1-\phi_2(\omega_2, M)) \quad (91) \\
\Pr[(M, \omega_2 + 1) \,|\, (M, \omega_2)] &= (1-\delta)\phi_2(\omega_2, M) \quad (92)
\end{aligned}$$

9. Suppose that (ω_1, ω_2) is such that $\omega_1 \notin \{0, M\}$ and $\omega_2 = M$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(M, \omega_1) \,|\, (\omega_1, M)] &= \delta\phi_2(M, \omega_1)\phi_1(\omega_1, M) \\
&\quad + (1-\delta)(1-\phi_1(\omega_1, M)) \quad (93) \\
\Pr[(\omega_1, M-1) \,|\, (\omega_1, M)] &= \delta(1-\phi_2(M, \omega_1))\phi_1(\omega_1, M) \quad (94) \\
\Pr[(\omega_1 - 1, M) \,|\, (\omega_1, M)] &= \delta\phi_2(M, \omega_1)(1-\phi_1(\omega_1, M)) \quad (95) \\
\Pr[(\omega_1 - 1, M-1) \,|\, (\omega_1, M)] &= \delta(1-\phi_2(M, \omega_1))(1-\phi_1(\omega_1, M)) \quad (96) \\
\Pr[(\omega_1 + 1, M) \,|\, (\omega_1, M)] &= (1-\delta)\phi_1(\omega_1, M). \quad (97)
\end{aligned}$$

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